進階代數(下) 第二次作業

上課老師: 翁志文

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The *order* $|a|$ of an element $a \in G$ is the cardinality of the subgroup $< a >$ of G. The *direct sum* $H \oplus G$ of two groups H and G is the group with set $H \times G$ and operation is defined componentwise inherited from H and G respectively.

1. Let G be the multiplicative group of all nonsingular 2×2 matrices with rational entries, and
$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$
   (a) (羅建峰) Show a has order 4 and b has order 3.
   (b) (陳泓勳) What is the order of ab.
2. (何昕暘) Let G be a group that has only a finite number of subgroups. Show that G is a finite group.
3. (賴德展) Let H, K, N be subgroups of a group G such that $H < K$, $H \cap N = K \cap N$, and $HN = KN$. Show that $H = K$.
4. (劉侖欣) Let N be a subgroup of index 2 in a group G. Show that N is normal in G.
5. Suppose $H < G$. The *normalizer* of H in G is defined to be the set
$$N_G(H) = \{a \in G \mid a^{-1}Ha = H\}.$$
   (a) (周彥伶) Show that $N_G(H)$ is a subgroup of G.
   (b) (洪湧昇) Show $H \lhd N_G(H)$.
6. (林志峰) Suppose that N and M are two normal subgroups of G and that $N \cap M = < e >$. Show that $nm = mn$ for $n \in N$ and $m \in M$.
7. (黃正一) Let H and K be subgroups of a group G. Show that
$$[< H, K >: H] \geq [K : H \cap K].$$
8. (邱鈺傑) Let G be a finite group and ϕ an automorphism of G such that ϕ^2 is the identity automorphism of G. Suppose that $\phi(x) = x$ implies that $x = e$. Prove that G is abelian and $\phi(a) = a^{-1}$.