第二次作業

5. (a) Take $a, b \in N_G(H)$.

   Claim: $ab^{-1} \in N_G(H)$

   (1) $a \in N_G(H) \Rightarrow a^{-1} \in N_G(H)$

   (i) Take $aha^{-1} \in aHa^{-1}$ where $h \in H = a^{-1}Ha$

   $\exists h'$ such that $h = a^{-1}h'a \Rightarrow aha^{-1} = a(a^{-1}h'a)a^{-1} = h' \in H$

   $\Rightarrow aHa^{-1} \subseteq H$.

   (ii) Take $h \in H, h = a\underbrace{a^{-1}ha}_{\in H}a^{-1} \in aHa^{-1} \Rightarrow H \subseteq aHa^{-1}$

   By (i), (ii) we have $a^{-1} \in N_G(H)$.

   (2) Take $a, b \in N_G(H)$

   $ba^{-1}Hab^{-1} = H$ since $a^{-1}Ha = H$ and $bHb^{-1} = H$

   $(ab^{-1})^{-1}H(ab^{-1}) = H \Rightarrow ab \in N_G(H)$

   (b) (1) $H < N_G(H) \Rightarrow \forall h \in H, h^{-1}Hh = H$

   $\Rightarrow H \subseteq N_G(H)$

   Then, $H < N_G(H)$ since $H < G$

   (2) $\forall a \in N_G(H)$ we have $a^{-1}Ha = H$ by definition

6. Consider $(nm)(mn)^{-1} = nmn^{-1}m^{-1}$

   (1) $nmn^{-1} \in M, m^{-1} \in M \Rightarrow (nm)(mn)^{-1} \in M$

   (2) $n \in N, mn^{-1}m^{-1} \in N \Rightarrow (nm)(mn)^{-1} \in N$

   Since $(nm)(mn)^{-1} \in N \cap M = \{e\}$

   $\Rightarrow (nm)(mn)^{-1} = e$

   Thus $nm = mn$

7. $\because K = (H \cap K)t_1 \dot\cup \cdots \dot\cup (H \cap K)t_s, s = [K : H \cap K]$

   $\Rightarrow HK = Ht_1 \dot\cup \cdots \dot\cup Ht_s$

   Since $< H, K > \supseteq HK, [< H, K >: H] \geq s = [K : H \cap K]$

8. Define $h(x) = x^{-1}\phi(x)$

(1) one to one, i.e. $h(c) = h(b) \Rightarrow c = b$

$$\begin{aligned} c^{-1}\phi(c) &= b^{-1}\phi(b) \\ b \cdot c^{-1} &= \phi(b)\cdot\phi(c^{-1}) \\ &= \phi(bc^{-1}) \end{aligned}$$

$\therefore bc^{-1} = e$

(2) onto

$$\begin{aligned} c &= d^{-1}\phi(d) \\ b &= e^{-1}\phi(e) \\ cb &= g^{-1}\phi(g) \text{ for some } d, e, g \in G \end{aligned}$$

$$\begin{aligned} [d^{-1}\phi(e)\cdot e^{-1}\phi(e)]^{-1} &= [g^{-1}\phi(g)]^{-1} \\ &\Rightarrow [e^{-1}\phi(e)]^{-1}[d^{-1}\phi(d)]^{-1} = \phi(g)^{-1}g \\ &\Rightarrow \phi(e^{-1})\cdot e\cdot\phi(d^{-1})\cdot d = \phi(g^{-1})g \\ &\Rightarrow \phi(e^{-1}\phi(e))\cdot\phi(d^{-1}\phi(d)) = \phi(g^{-1}\phi(g)) \\ &\Rightarrow \phi(b)\cdot\phi(c) = \phi(cb) \\ &\Rightarrow \phi(bc)\phi(cb) \\ &\Rightarrow bc = cb \end{aligned}$$

$$\begin{aligned} \phi(a\cdot\phi(a)) &= \phi(a)\cdot\phi(\phi(a)) \\ &= \phi(a)\cdot a = a\cdot\phi(a) \\ &\Rightarrow a\cdot\phi(a) = e\ (\phi(a\cdot\phi(a)) = \phi(a) \\ &\Rightarrow \phi(a) = a^{-1} \end{aligned}$$