第五次作業

1. $H < G \Rightarrow G \neq \bigcup_{g\in G} gHg^{-1}$

   $N_G(H) = \{g \in G | gHg^{-1} = H\}$

   $G \diagup_{N_G(H)} = \{g_1 N_G(H), g_2 N_G(H), \cdots, g_k N_G(H)\}$

Case i

$$\begin{aligned} k = 1 &\Rightarrow G = N_G(H), \\ &\Rightarrow \forall g \in G,\ gHg^{-1} = H \\ &\Rightarrow N \lhd G \\ &\Rightarrow \bigcup_{g\in G} gHg^{-1} = H < G \\ &\Rightarrow G \neq \bigcup_{g\in G} gHg^{-1} \end{aligned}$$

Case ii $k \geq 2$. Let $a, b \in g_i N_G(H)$

$\Rightarrow a = bh, \forall h \in N_G(H)$

Then we have

$$\begin{aligned} aN_G(H)a^{-1} &= bhN_G(H)(bh)^{-1} \\ &= bhN_G(H)h^{-1}b^{-1} (\text{since } h \in N_G(H)) \\ &= bN_G(H)b^{-1} \end{aligned}$$

and,

$$\begin{aligned} |\bigcup_{g\in G} gHg^{-1}| &= |\bigcup_{i=1} g_i H g_i^{-1}| \\ &\leq \sum_{i=1}^{k} |g_i H g_i^{-1}| \\ &= k \cdot |H| \\ &\leq |G| \end{aligned}$$

2. (a) Let $U =< \{g^{-1}yg | g \in F(X), y \in Y\} >$ and $N = \bigcap_{Y \leq N' \lhd F(X)} N'$

Claim: $N = U$

" $\supseteq$ "

Claim: $U \supseteq N$ i.e. Claim: $U \lhd F(X)$ and $Y \subseteq U$

Since $U =< \{g^{-1}yg | g \in F(X), y \in Y\} >$.

Let $u = g_1^{-1}y_2g_1g_2^{-1}y_2g_2 \cdots g_k^{-1}y_kg_k \in U, g_i \in F(X), i = 1, 2, \cdots, k$.

$$
\begin{aligned}
x^{-1}ux &= x_{-1}g_1^{-1}y_2g_1g_2^{-1}y_2g_2 \cdots g_k^{-1}y_kg_kx \\
&= x_{-1}g_1^{-1}y_2g_1xx^{-1}g_2^{-1}y_2g_2xx^{-1} \cdots xx^{-1}g_k^{-1}y_kg_kx \\
&= [(g_1x)^{-1}y_2(g_1x)][(g_2x)^{-1}y_2(g_2x)] \cdots [(g_kx)^{-1}y_k(g_kx)] \ \in U \\
& \quad (\text{since } g_i \in F(X), i \in \{1, 2, \cdots, k\}, \\
& \quad x \in F(X), F(X) \text{is group} \Rightarrow g_ix \in F(X)) \\
\Rightarrow & \quad x^{-1}ux \in U \\
\Rightarrow & \quad U \lhd F(X) \text{and} \forall y \in Y, y = e^{-1}ye \in U, 1 = e^{-1} \in F(X) \\
\Rightarrow & \quad Y \subseteq U
\end{aligned}
$$

Hence, $U \supseteq N$

" $\subseteq$ "

Claim: $U \subseteq N$

Let $u \in U, u = g_1^{-1}y_2g_1g_2^{-1}y_2g_2 \cdots g_k^{-1}y_kg_k \in U, g_i \in F(X), i = 1, 2, \cdots, k$.

$\because N' \lhd F(X)$ and Y

$subseteq N'$

$\therefore \forall y \in Y, g^{-1}yg \in N', \forall g \in F(X)$

$\therefore u = (g_1^{-1}y_2g_1)(g_2^{-1}y_2g_2) \cdots (g_k^{-1}y_kg_k) \in N' \ \forall N'$

We have $u \in \bigcap_{Y \leq N' \lhd F(X)} N' = N \Rightarrow U \subseteq N$.

(b) $X = \{x, y\}, Y = \{y\}$.

Claim: $F(X) \diagup_{N_Y} \cong F(\{x\}), F(\{x\}) = \{x^i | i \in \mathbb{Z}\}$.

Define: $\phi : F(X) \to F(\{x\})$ by $\phi(x) = x, \phi(y) = e$ and

$\phi(x^{s_1}y^{t_1}x^{s_2}y^{t_2} \cdots x^{s_n}y^{t_n}) = x^{\sum_{i=1}^{n} s_i}$

Claim: $N = ker\phi = \{h \in F(X) | \phi(h) = e\}$.

" $\subseteq$ "

Since $\forall n \in N, \exists g \in F(X)$ such that $n = g_1^{-1} y g_1 g_2^{-1} y g_2 \cdots g_k^{-1} y g_k$

$$\begin{aligned}\phi(n) &= \phi(g_1^{-1} y g_1 g_2^{-1} y g_2 \cdots g_k^{-1} y g_k) \\ &= \phi(g_1^{-1})\phi(g_1)\phi(g_2^{-1})\phi(g_2) \cdots \phi(g_k^{-1})\phi(g_k) \\ &= e\end{aligned}$$

$\Rightarrow n \in ker\phi$
$\Rightarrow N \subseteq ker\phi$
" $\supseteq$ "
Claim: $ker\phi \subseteq N$ i.e. Claim: $ker\phi \backslash N \subseteq N$
Let $x^{s_1} y^{t_1} x^{s_2} y^{t_2} \cdots x^{s_n} y^{t_n} \in ker\phi \backslash N$ and by (a) $x^{s_1} y^{-t_1} x^{-s_1} \in N$
$\Rightarrow x^{s_1} y^{-t_1} x^{-s_1} x^{s_1} y^{t_1} x^{s_2} y^{t_2} \cdots x^{s_n} y^{t_n} \in ker\phi \backslash N$
$\Rightarrow x^{(s_1+s_2)} y^{t_1} x^{s_2} y^{t_2} \cdots x^{s_n} y^{t_n} \in ker\phi \backslash N$
$\Rightarrow x^{(s_1+s_2)} y^{-t_2} x^{-(s_1+s_2)} x^{(s_1+s_2)} y^{t_1} x^{s_2} y^{t_2} \cdots x^{s_n} y^{t_n} \in ker\phi \backslash N$
$\vdots$
$\Rightarrow x^{\sum s_i} \in ker\phi \backslash N$
Since $x^{\sum s_i} \in ker\phi$ $(\phi(x^{\sum s_i}) = x^{\sum s_i} = 1 = x^0)$
We have $\sum_{i=1}^{n} s_i = 0$ and $e \in ker\phi \backslash N$ which contradict to $N \lhd G$
$\Rightarrow ker\phi = N$.
By 1-st homomorphism, $F(X) \diagup_{N_Y} \cong F(\{x\})$.

3. (a) Define $\phi: F(X) \to L$ by $\phi(s_1) = A_1, \phi(s_2) = A_2, \phi(s_3) = A_3$.

Define $\psi: W = F(X)\diagup_{N_Y} \to L$ by $aF_Y \to \phi(a), a \in F(X)$

Check that ψ is "well-defined" and "homomorphism".

"well-defined"

Suppose $aN_Y = bN_Y \Rightarrow b^{-1}a \in N_Y$.

Claim: $\psi(aN_Y) = \psi(bN_Y)$.

$\phi(b^{-1}a) = \psi(b^{-1}aN_Y) = \psi(N_Y) = \phi(e)$

$\because \phi(b^{-1}a) = \phi(b)^{-1}\phi(a) = \phi(e) = e$

$\therefore \phi(a) = \phi(b)$ i.e. $\psi(aN_Y) = \psi(bN_Y)$.

"homomorphism"

$$\begin{aligned}\psi((aN_Y)(bN_Y)) &= \psi(abN_Y) \\ &= \phi(ab) \\ &= \phi(a)\phi(b) \\ &= \psi(aN_Y)\psi(bN_Y)\end{aligned}$$

Finally, we check that

$$\begin{aligned}(A_1A_3)^2 &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mod 2 \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \mathrm{I}\end{aligned}$$

And check others similarly.

(b) $P_1 = \{1\}, \therefore W =< s_1 >$. Let $X = \{s_1\}, Y = \{(s_1s_1)\} = \{s_1^2\}$, and $(s_1s_1)^{m(s_1,s_1)} = (s_1s_1)^1 = e$

$\therefore$ Coxeter group $W = F(X)\diagup_{N_Y} = \{s_1, e\}$

Claim: $W \cong \mathbb{Z}_2$, Define: $\phi : F(X) \to \mathbb{Z}_2$ by $\phi(s_1) = 1, \phi(e) = 0$ (or $\phi(s_1^2) = 0$)

Claim: $ker\phi = N_Y$

" $\supseteq$ "

Let $k \in N_Y \Rightarrow k = g_1 y g_1^{-1} g_2 y g_2^{-1} \cdots g_h y g_h^{-1} \in F(X)$

where $g_i \in F(X), i \in \{1, \cdots, h\}, y \in Y = \{s_1^2\}$

$$\begin{aligned}
\therefore \phi(k) &= \phi(g_1 y g_1^{-1} g_2 y g_2^{-1} \cdots g_h y g_h^{-1}) \\
&= \phi(g_1)\phi(y)\phi(g_1^{-1})\phi(g_2)\phi(y)\phi(g_2^{-1})\phi \cdots \phi(g_h)\phi(y)\phi(g_h^{-1}),\ y = s_1^2 \\
&= \phi(g_1)\phi(g_1^{-1})\phi(g_2)\phi(g_2^{-1})\phi \cdots \phi(g_h)\phi(g_h^{-1}) \\
&\quad \because \phi(s_1^2) = \phi(s_1 s_1) = \phi(s_1) + \phi(s_1) = 1 + 1 = 0 = \phi(y) \\
&= 0
\end{aligned}$$

$\therefore k \in ker\phi \Rightarrow N_Y \subseteq ker\phi.$

" $\subseteq$ "

Claim: $ker\phi \subseteq N_Y$ i.e. Claim: $ker\phi \backslash N_Y \subseteq N_Y$.

Let $s_1^{t_1} \in ker\phi \backslash N_Y$

$$\begin{aligned}
\phi(s_1^{t_1}) &= t_1\ (\because \phi(s_1) = 1 \therefore \phi(s_1^{t_1}) = \phi(s_1) + \cdots + \phi(s_1)) \\
&= 0\ (\because s_a^{t_1} \in ker\phi \therefore \phi(s_1^{t_1}) = 0) \\
&\therefore\ t_1 = 0 \\
&\therefore\ s_1^{t_1} = e (\text{which contradict to } N_Y \text{ is normal subgroup})
\end{aligned}$$

We have $ker\phi = N_Y$

(c) P_2 i.e. $X = \{s_1, s_2\}$

(1) if $i = j \Rightarrow s_1^2 = s_2^2 = e$

(2) if i, j are adjacent

$$\Rightarrow (s_1 s_2)^{m(s_1,s_2)} = (s_1 s_2)^3 = e \text{ and } (s_2 s_1)^{m(s_2,s_1)} = (s_2 s_1)^3 = e$$

(3)

$$(s_1 s_2 s_1)(s_2 s_1 s_2) = s_1 s_2^2 s_1 = s_1^2 = e, (s_2 s_1 s_2)(s_2 s_1 s_2) = e$$

(4) Check that $s_1s_2s_1 = s_2s_1s_2$,

$$\begin{aligned} s_1s_2 &= s_1(s_1s_2s_1s_2s_1s_2)s_2 = \\ &\Rightarrow s_1s_2 = s_2s_1s_2s_1 \\ &\Rightarrow s_2s_1s_2 = s_2s_2s_1s_2s_1 \\ &\Rightarrow s_2s_1s_2 = s_1s_2s_1 \end{aligned}$$

$\therefore F(X)\diagup_{N_Y} = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \cong D_3 = \{\sigma^i\tau^j | i = 0, 1, 2, j = 0, 1\}$

$X = \{s_1, s_2\}, Y = \{s_1^2, s_2^2, (s_1s_2)^3, (s_2s_1)^3, (s_1s_2s_1)^2\}$

Claim: Coxeter group $F(\{s_1, s_2\})\diagup_{N_Y} \cong D_3$

Define: $\phi : F(\{s_1, s_2\}) \to D_3$ by $\phi(s_1) = \tau, \phi(s_2) = \tau\sigma$ (since $(\tau\sigma)^2 = \tau\sigma\tau\sigma = \tau\sigma\sigma^{-1}\tau = \tau^2 = e$)

Claim: $ker\phi = N_Y$, where $Y = \{s_1^2, s_2^2, (s_1s_2)^3, (s_2s_1)^3, (s_1s_2s_1)^2\}$

The way is as same as (b)

(d)

$$L_1 = \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ x+y \end{bmatrix} \therefore L_1\begin{pmatrix} x \\ y \end{pmatrix} = A_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ x+y \end{pmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x+y \\ y \end{bmatrix} \therefore L_2\begin{pmatrix} x \\ y \end{pmatrix} = A_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}, A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$\therefore A_1^2 = A_2^2 = I$

Define $\phi : F(X) \to L$ by $\phi(s_1) = A_1, \phi(s_2) = A_2$

$L =< \{A_1, A_2\} >$

Since $A_1^2 = I = A_2^2$ then A_1^{-1}, A_2^{-1} exist, and $A_1^{-1} = A_1, A_2^{-1} = A_2$

And $I \in L$

L is a group.

$L = \{I, A_1, A_2, (A_1A_2), (A_2A_1), A_1A_2A_1 = A_2A_1A_2\} \because (A_1A_2)^3 = e = (A_2A_1)^3$

$\therefore (A_1A_2)^2 = (A_2A_1)(A_1A_2)^3 = (A_2A_1)e = A_2A_1$

$\Rightarrow A_1A_2A_1A_2 = A_2A_1 \Rightarrow A_2A_1A_2 = A_1A_2A_1$