Solution for Homework 6 part 1, problem 1, 2, 4 and 5
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1. *Show that no group is the union of its two proper subgroups.*

Suppose $G = N \cup H$ where N, H are proper subgroups of G.
If one of N, H contains the other, then clearly we have a contradiction that $G \neq N \cup H$.
Thus we may assume there exist n, h with $n \in N \backslash H$ and $h \in H \backslash N$, then consider nh. $nh \notin N \cup H$, since if $nh \in N$ we have $n^{-1}nh = h \in N$, a contradiction. Similar for $nh \in H$. Thus we are done. ■

2. *Find an abelian group G and a subgroup H of G such that $G \neq H \times K$ for all subgroups K of G.*

Let $G = \mathbb{Z}$, $H = 2\mathbb{Z}$. Recall by definition $G = H \times K$ implies that H, $K \lhd G$, $G = HK$ and $H \cap K = e$. The last property shows there is no subgroup K of G such that $G = H \times K$, since for any $k \in K$, $2k \in 2\mathbb{Z} = H$. ■

4. *Show that the additive group $\mathbb{Q}$ is indecomposable.*

Suppose not. Then we have $Q = A \times B$ for some A, $B \lhd G$. Pick $e \neq \frac{a_1}{a_2} \in A$ and $e \neq \frac{b_1}{b_2} \in B$, then $\frac{a_1}{a_2} + \frac{b_1}{b_2} = \frac{a_1b_2+a_2b_1}{a_2b_2} \in A \cap B = \{e\} = \{0\}$. This implies $a_1b_2 + a_2b_1 = 0$, therefore $a_1b_2 = -a_2b_1$ and $\frac{a_1}{a_2} = -\frac{b_1}{b_2} \in A \cap B = \{e\}$, a contradiction. ■

5. *Write the additive group $\mathbb{Z}_{150}$ as a finite direct sum of indecomposable subgroups.*

We claim that $\mathbb{Z}_{150} =< 75 > \times < 50 > \times < 6 >$. To prove it we need to check

(i) $< 75 >, < 50 >, < 6 > \lhd \mathbb{Z}_{150}$
It's clear that those 3 are subgroups of $\mathbb{Z}_{150}$. Since $\mathbb{Z}_{150}$ is abelian, thus they are normal.

(ii) $\mathbb{Z}_{150} =< 75 >< 50 >< 6 >$
Since $gcd(75, 50, 6) = 1$ there exist integers a, b, c such that $75a + 50b + 6c = 1$. Therefore $\mathbb{Z}_{150} =< 75 >< 50 >< 6 >$.

(iii) $< 75 > \cap < 50 >< 6 >=< 50 > \cap < 75 >< 6 >=< 6 > \cap < 75 >< 50 >= \{e\}$
Note that $< 75 >\cong \mathbb{Z}_2$, $< 50 >\cong \mathbb{Z}_3$ and $< 6 >\cong \mathbb{Z}_{25}$.
And $\mathbb{Z}_3\mathbb{Z}_{25}$ has no subgroup of order 2, $\mathbb{Z}_3\mathbb{Z}_2$ has no subgroup of order 25, $\mathbb{Z}_{25}\mathbb{Z}_2$ has no subgroup of order 3. Thus we are done. (Or one can use the following technique: $75 \neq 50a + 6b$ for any integer a, b, since 2 divides RHS but not LHS. Rest are similar.) ■

Comment from teacher:
In $\mathbb{Z}_{150} = <75> \times <50> \times <6>$ we can replace $<6>$ with any subgroup of order 6, like $<12>$. Similarly we can replace $<50>$ by $<100>$.