Homework 9

1.

$f : \mathbb{Z} \mapsto \mathbb{Z}_6 =< 2 > \times < 3 >$

by $f(x) = r$, where $r \equiv x \pmod 6$

(1)homomorphism:

$f(x + y) \equiv x + y \pmod 6 = f(x) + f(y)$

(2)onto:

choose $\{0, 1, 2, 3, 4, 5\} \in \mathbb{Z}$, done.

2.

(a) subgroup of A_4:
$\{e\}$,{ e, (12)(34), (13)(24), (14)(23) }, A_4
if A_4 is decomposable
$\Rightarrow A_4 = H \times K$, and H, K are nontrivial normal subgroups.
But we can't find them, so A_4 is indecomposable.

(b) $N = \langle(1, 1)\rangle$, $N = \langle(1, 0)\rangle$, $K = \langle(0, 1)\rangle$
$\langle(1, 1)\rangle \subseteq \langle(1, 0)\rangle + \langle(0, 1)\rangle$
$\langle(1, 1)\rangle \cap (\langle(1, 0)\rangle + \{(0, 0)\})$=$\{(0, 0)\}$
$\langle(1, 1)\rangle \cap (\{(0, 0)\} + \langle(0, 1)\rangle)$=$\{(0, 0)\}$

(c) Suppose $N \cap H = \{e\}$, $N \cap K = \{e\}$
Since N, H, K is normal in G
and $nhn^{-1}h^{-1} = (nhn^{-1})h^{-1} \in H$
$= n(hn^{-1}h^{-1}) \in N$
$= e(H \cap N = \{e\})$
$\Rightarrow nh = hn$, similarly $nk = kn$
Consider $g = hk \in G$
$gn = hkn = hnk = nhk = ng$ for all $n \in N, g \in G$
$\Rightarrow N < Z(G)$

(d) S_2, S_3 is indecomposable, it's clear.
Suppose not for $n \geq 5$
Let $S_n = H \times K$
Note: $A_n \lhd S_n$ (since $(12)(34) \in A_n$ and $(13) \in S_n$)
But $(12)(34)(13) = (1432)$ and $(13)(12)(34) = (1234)$,so $A_n \not< S_n$
If $A_n \cap H \neq \{e\}$ hold,
$\because A_n \lhd S_n$ and $H \lhd S_n \Rightarrow A_n \cap H \lhd A_n (\rightarrow\leftarrow)$
(By Theorem 1.5.3, $K < G$, $N \lhd G$, $\Rightarrow N \cap K \lhd K$)
Similarly, $A_n \cap K = \{e\}$.
Contradiction to (c), thus S_n is indecompasoble for $n \geq 5$.
The normal subgroup of S_4 is $\{e\}$, K_4, A_4, S_4.
Suppose S_n is decomposable,
$\Rightarrow S_n = K_4 \times A_4$, but $K_4 \cap A_4 \neq \{e\} (\rightarrow\leftarrow)$
$\therefore S_n$ is indecomposable.

3. G is a finite group,

$* : G \times G \to G, g * a = ga \ \forall a \in G$

$\phi : G \to S_G$ is a homomorphism.

$G' \cong G$, $G \cap G' = \phi$

$*' : G \times (G' \cup G) \to G' \cup G$

$$g *' a = \begin{cases} ga & \text{if } a \in G, \\ g'a & \text{if } a \in G' \end{cases}$$

$\phi' : G \to S_{G \cup G'}$, $ker\phi' = \{e\}$
$\Rightarrow G \cong Im\phi'$
$g \to c_1 \cdot c_2$, c_1, c_2 are both even or both odd.
$\therefore G \cong A_{2|G|}$

4.
$G/H = \{eH, a_1H, a_2H, \cdots, a_{m-1}H\}$, $|G/H| = m$
Define: $\sigma g : G/H \to G/H$ by $g(aH) = (ga)H$
$\phi : G \to S_m$ by $g \to \sigma g$, $ker\phi \lhd G$
$G/ker\phi \cong Img\phi \leq S_m$, $|G/ker\phi| \leq m!$.

5.
Let $G/H = \{eH, a_1H, a_2H, \cdots, a_{n-1}H\}$, $|G/H| = n$
Define: $\sigma g : G/H \to G/H$ by $g(aH) = (ga)H$
$\phi : G \to S_m$ by $g \to \sigma g$, $ker\phi \lhd G$
$n = |G|/|H|$, let $K = ker\phi$
For $k \in K$, $keH = eH$, $\Rightarrow K \subseteq H$
Hence $n = \dfrac{|G|}{|H|} \mid \dfrac{|G|}{|K|} = |Img\phi| \mid |S_n| = n!$
$\Rightarrow n \mid \dfrac{|G|}{|H|}$, and $\dfrac{pn}{|K|} \mid n!$
$\Rightarrow |K| = p$
Since $|K| = |H|$, $K \subseteq H \Rightarrow K = H = ker\phi \lhd G$

6.
Let $S = N$ and G acts on N by conjugation
i.e. $gn = gng^{-1}$ for all $g \in G$, $n \in N$
$p = |N| = |S| = \sum_i |O_{s_i}| = \sum_{|O_{s_i}|=1} 1 + \sum_{|O_{s_i}| \neq 1} \dfrac{|G|}{|G_{s_i}|}$
where $|O_{s_i}| = 1 \Leftrightarrow G = G_{s_i}$
$\Leftrightarrow g \cdot s_i = gs_ig^{-1} = s_i \; \forall g \in G$
$\Leftrightarrow s_i \in Z(G) \cap N$
$\therefore p = |Z(G) \cap N| + \sum_{|O_{s_i}| \neq 1} |O_{s_i}|$
$= |z(G) \cap N| + \sum_{|G_{s_i}| \neq |G|} \dfrac{|G|}{|G_{s_i}|}$
$\because p | \sum \dfrac{|G|}{|G_{s_i}|}$ and $|Z(G) \cap N| \neq 0$
$\therefore \sum \dfrac{|G|}{|G_{s_i}|} = 0$
$\Rightarrow Z(G) \cap N = N$
$\Rightarrow N \subseteq Z(G)$