Homework 6

2009.3.26

5.

(a) $\overline{0} = \overline{0/p^i}$ so $\overline{0} \in \mathbb{Z}(p^\infty)$ and $\mathbb{Z}(p^\infty)$ must not be empty. Let $\overline{a/p^i}$ and $\overline{b/p^j}$ be elements in $\mathbb{Z}(p^\infty)$, the addition $\overline{a/p^i} + \overline{b/p^j} = \overline{\frac{ap^j+bp^i}{p^{ij}}} \in \mathbb{Z}(p^\infty)$ is closed. Associativity holds because $\mathbb{Q}/\mathbb{Z}$ is an additive group. Lastly, $-\overline{a/p^i} = \overline{-a/p^i}$ is also included in $\mathbb{Z}(p^\infty)$. Therefore, $\mathbb{Z}(p^\infty)$ is a subgroup of $\mathbb{Q}/\mathbb{Z}$.

(b) For any $\overline{a/p^i} \in \mathbb{Z}(p^\infty)$, we have

$$\underbrace{\overline{a/p^i} + \overline{a/p^i} + \ldots + \overline{a/p^i}}_{p^i} = p^i\overline{a/p^i} = \overline{p^i a/p^i} = \overline{a} = \overline{0}.$$

Hence $\overline{a/p^i}$ is of order $n \leqslant p^i$.

(c) Let H be a subgroup of $\mathbb{Z}(p^\infty)$. By (b) we know every element of H is of finite order p^n for some n. Suppose at least one element of H has order p^i and no element of H has order greater than p^i. It can be easily seen that $\overline{1/p^i}$ is one of such elements and $< \overline{1/p^i} > \subseteq H$. Then for any element $\overline{a/p^j} \in H$ where $j \leqslant i$,

$$ap^{i-j}\overline{1/p^i} = \overline{ap^{i-j}/p^i} = \overline{a/p^j} \in < \overline{1/p^i} > .$$

The end result is that H is the cyclic subgroup generated by $\overline{1/p^i}$. Suppose H has no such element, i.e. there is no upper bound on the orders of elements of H, we will show that $H = \mathbb{Z}(p^\infty)$. Obviously, $H \subseteq \mathbb{Z}(p^\infty)$. Select an element $\overline{a/p^i} \in \mathbb{Z}(p^\infty)$, as above shows, $\overline{a/p^i} \in\ <\overline{1/p^i}>$. Since there is no upper bound on the orders of elements of H, $<\overline{i/p^i}> \subset H$. Therefore, $H = \mathbb{Z}(p^\infty)$.

(d) By (c), every proper subgroup of $\mathbb{Z}(p^\infty)$ is a cyclic group generated by $\overline{a/p^k}$ for some k. Consider a chain $G_1 > G_2 > \ldots$ of subgroups of $\mathbb{Z}(p^\infty)$. If $G_r \neq \mathbb{Z}(p^\infty)$ for some r, choose n such that G_n has smallest order, then $G_i = G_n$ for all $i \geqslant n$. So $\mathbb{Z}(p^\infty)$ satisfies ACCN. For the chain $\overline{1/p^0} < \overline{1/p^1} < \overline{a/p^2} \ldots$, there is no integer n such that $G_i = G_n$ for all $i \geqslant n$. This shows that $\mathbb{Z}(p^\infty)$ doesn't satisfy DCCN.

(e) Suppose $\mathbb{Z}(p^\infty) = <\overline{1/p^{i_1}}> \times <\overline{1/p^{i_2}}> \times \ldots \times <\overline{1/p^{i_2}}>$. But $<\overline{1/p^{i_j}}> \cap <\overline{1/p^{i_k}}> \neq \emptyset$ when $i_j < i_k$, this is a contradiction. Therefore, $\mathbb{Z}(p^\infty)$ is not the (internal) direct product of two of its proper subgroups, i.e. $\mathbb{Z}(p^\infty)$ itself is indecomposable.