6.

(a)

(1,3)=(1,2)(2,3)(1,2)

(1,4)=(1,3)(3,4)(1,3)

.

.

.

(h,k)=(h,k-1)(k-1,k)(h,k) ,

where 1 $\leqslant$ h $\leqslant$ k-1 , 2 $\leqslant$k $\leqslant$h+1

so (k-2,k-1)...(h+1,h+2)(h,h+1)(h,k)(h,h+1)(h+1,h+2)...(k-2,k-1) = (k-1,k)

thus, (h,k)=(h,h+1)...(k-2,k-1)...(h,h+1) $\square$

(b)

$W =< S_1, S_2, ..., S_n >$ with $\begin{cases} (S_iS_{i+1})^3 = e = (S_{j+1}S_i)^3 \\ S_i^2 = e \\ (S_iS_j)^2 = e = (S_jS_i)^2 \ if \ j \neq i+1 \end{cases}$

$L =< S_1, S_2, ..., S_n >$

Let $f : W \longrightarrow L$ by $f(S_i) = S_i$

Check

(1)$(S_iS_{i+1})^3 = I = (S_{i+1}S_i)^3$

(2)$S_i^2 = I$

(3)$(S_iS_j)^2 = I = (S_jS_i) \ if \ j \neq i+1$

$$S_i = \begin{bmatrix} I & & \cdots & & \\ & 1 & 1 & 0 & \\ \vdots & 0 & 1 & 0 & \vdots \\ & 0 & 1 & 1 & \\ & & \cdots & & I \end{bmatrix}$$

the ith row

(1)

$$(S_iS_{i+1}) = \begin{bmatrix} I & & & \cdots & & \\ & 1 & 1 & 0 & 0 & \\ \vdots & 0 & 1 & 0 & 0 & \vdots \\ & 0 & 1 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & \\ & & \cdots & & & I \end{bmatrix} . \begin{bmatrix} I & & & \cdots & & \\ & 1 & 0 & 0 & 0 & \\ \vdots & 0 & 1 & 1 & 0 & \vdots \\ & 0 & 0 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & \\ & & \cdots & & & I \end{bmatrix} = \begin{bmatrix} I & & & \cdots & & \\ & 1 & 1 & 1 & 0 & \\ \vdots & 0 & 1 & 1 & 0 & \vdots \\ & 0 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 1 & \\ & & \cdots & & & I \end{bmatrix}$$

then ,$(S_iS_{i+1})^3 = I$ $\square$

(2)

$$S_i^2 = \begin{bmatrix} I & & \cdots & & \\ & 1 & 1 & 0 & \\ \vdots & 0 & 1 & 0 & \vdots \\ & 0 & 1 & 1 & \\ & & \cdots & & I \end{bmatrix} \cdot \begin{bmatrix} I & & \cdots & & \\ & 1 & 1 & 0 & \\ \vdots & 0 & 1 & 0 & \vdots \\ & 0 & 1 & 1 & \\ & & \cdots & & I \end{bmatrix} = I \ \square$$

(3)

$$(S_iS_j) = \begin{bmatrix} I & & & \cdots & & & \\ & 1 & 1 & 0 & 0 & 0 & \\ \vdots & 0 & 1 & 0 & 0 & 0 & \vdots \\ & 0 & 1 & 1 & 0 & 0 & \\ & 0 & 0 & 0 & 1 & 0 & \\ & 0 & 0 & 0 & 0 & 1 & \\ & & \cdots & & & & I \end{bmatrix} \cdot \begin{bmatrix} I & & & \cdots & & & \\ & 1 & 0 & 0 & 0 & 0 & \\ \vdots & 0 & 1 & 0 & 0 & 0 & \vdots \\ & 0 & 0 & 1 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & 0 & \\ & 0 & 0 & 0 & 1 & 1 & \\ & & \cdots & & & & I \end{bmatrix}$$

then $(S_iS_j)^2 = I_n$ and for $j > i+2$ $S_iS_j = SjS_i$
$(S_iS_j)^2 = (S_iS_j)(S_jS_i) = [(S_iS_j)S_j]S_i = [S_i(S_jS_i)]S_i = S_iS_i = I$ $\square$

(c)
Define L act on S by $g \circ s = gs$
This satisfies Axiom1,2
But we need to check that $gs \in S$,it suffices to assume $g = S_i$
Note that

$$S_i \circledast a_j = \begin{cases} a_{j+1}, if\ j = i \\ a_{j-1}, if\ j = i+1 \\ a_j, otherwise \end{cases} \quad \square$$

(d)
By (c) we define $f_g : S \longrightarrow S$
such that $f_g(s) = gs$ for each $g \in L$ and $s \in S$
Then $\phi : L \longrightarrow S_{n+1}, g \longrightarrow fg$ is a homomorphism. $\square$

(e)
$X = \{S_1, S_2, S_3\}$
$Y = \{S_1^2, S_2^2, S_3^2, (S_1S_2)^3, (S_2S_1)^3, (S_2S_3)^3, (S_3S_2)^3, (S_1S_3)^3, (S_3S_1)^3\}$
$\phi : F(X) \longrightarrow S_4$ define by $S_1 \longrightarrow (1,2), S_2 \longrightarrow (2,3), S_3 \longrightarrow (3,4)$
ϕ is onto by (a),
claim that $ker\phi = N_y$
$N_y \subseteq ker\phi istrivial.$
On the other hand,assume not ,there is a element $u \in ker\phi - N_y$
By claim $W = F(X)/N_y \cong S_4$
Then it's down. $\square$