Homework 3.

1.(a)

Let $[F:K]=p$, p is a prime.

If L is a field between F and K,

then $p=[F:K]=[F:L][L:K]$.

Since p is a prime, one of $[F:L],[L:K]$ is 1, then $L=F$ or $L=K$.

If $[F:L]=1$ then F is a one-dimensional vector space over L, and the span of $1\in F$

$\Rightarrow L\cdot 1=F$ and $1\in L$ too, then $F=L$

(b)

(i)Claim: $[K(u):K]$ is odd $\Rightarrow [K(u^2):K]$ is odd.

pf: $K\subseteq K(u^2)\subseteq K(u)$

$\Rightarrow [K(u):K]=[K(u):K(u^2)][K(u^2):K]$

(ii)Claim: $K(u)=K(u^2)$

($\supseteq$) Trivial.

($\subseteq$) u satisfies $f(x)=x^2-u^2=0$ over $K(u^2)[x]$

If this polynomial is irreducible, we have $[K(u):K(u^2)]=2(\rightarrow\leftarrow)$

It is reducible over $K(u^2)\Rightarrow u\in K(u^2)$.

(c)

(i)Define a polynomial $f(y)=y^3-uy-u\in K(u)[y]$

$$f(x)=x^3-ux-u=x^3-\frac{x^3}{x+1}x-\frac{x^3}{x+1}=\frac{x^3}{x+1}((x+1)-x-1)=0$$

Hence $x\in K(x)$ is a root of polynomial $f\in K(u)[y]$

$\therefore x$ is algebraic over $K(u)$

Thus $K(x)$ is a extension of $K(u)$.

(ii)Claim: f is irreducible over $K(u)$

Suppose not, let $g(u)$ and $h(u)$ be relatively prime in $K[u]$ s.t $f(\frac{g(u)}{h(u)})=0$

Then $g(u)^3-ug(u)h(u)^2-uh(u)^3=0$

$\therefore g(u)$ and $h(u)$ have a nontrivial common factor. Thus $h(u)\in K$

$0=f(g(u))=f\cdot g\frac{x^3}{x+1}(\rightarrow\leftarrow$,$x$ is algebraic over $K)$

2.(a)

$\alpha,\beta\in(\alpha,\beta)$, $\alpha+\beta\in Q(\alpha,\beta)$, $\alpha\beta\in Q(\alpha,\beta)$, $[Q(\alpha,\beta):Q]=n$

$\{1,\alpha+\beta,(\alpha+\beta)^2,\cdots,(\alpha+\beta)^n\}$ and $\{1,\alpha\beta,(\alpha+\beta)^2,\cdots,(\alpha+\beta)^n\}$

Hence there are $a_i,b_j\in Q$, not all zero,

such that $\sum_{i=0}^n a_i(\alpha+\beta)^i=0$ and $\sum_{j=0}^n b_j(\alpha\beta)^j=0$

$\mathbb{Z}[x]$ is UFD.

For any $r\in Q(\alpha,\beta)$, r is algebraic number.

(b)

Let $x=\frac{3\sqrt{3}}{2}$

Then x is a root of $f(x)=8x^3-3$ irreducible over Q.

If x is an algebraic integer, then x is a root of a monic polynomial in $\mathbb{Z}[x]$

$x^n+a_{n-1}x^{n-1}+\cdots+a_0=f(x)g(x)$

But the leading coefficient of $f(x)$ is 8.$(\rightarrow\leftarrow)$

(c)

Let $n=0$, then $nu=0$ is an algebraic integer.

Suppose $n>0$, u is algebraic number.

$\Rightarrow f(u)=0$, for some $f(x)\in\mathbb{Z}[x]$

Assume c is the leading coefficient of f.

$f(x)=cx^n+\cdots,f(u)=0$

$c^{n-1}f(x)=(cx)^n+b_1(cx)^{n-1}+cb_2(cx)^{n-2}+\cdots$

$0 = c^{n-1} f(u), g(x) = x^n + b_1 x^{n-1} + b_2 x^{n-2} + \cdots$
$\Rightarrow g(cu) = 0 \Rightarrow cu$ is algebraic integer.