進階代數(下) 第一次作業解答

2009.3.2

1. Define a relation $a \backsim b$ in a group G if $a = b$ or $a = b^{-1}$. To show that $\backsim$ forms a partition, we examine reflexivity, symmetry and transitivity. Let $\Pi = \{P_1, P_2, \cdots, P_t\}$.
   Claim: $|P_i| \leq 2$.
   pf: it's clear by definition of the relation $\backsim$.
   Since G is a group of even order and $P_i = \{e\}$ for some i, there exists $P_j = \{a\}$ such that $a = a^{-1}$.

2. Let $\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \cdots, \overline{P-1}\}$. Clearly, $\mathbb{Z}_p$ is a monoid.
   Claim: $\forall a \in \mathbb{Z}_p - \{\overline{0}\}, \exists b \in \mathbb{Z}_p$, s.t. $ab \equiv \overline{1}$ (mod P).
   We know $\forall a, b \in \mathbb{Z}$, if $(a, b) = 1$, then $\exists x, y \in \mathbb{Z}$, s.t. $ax + by = 1$.
   Let $a \in \mathbb{Z}_p - \{\overline{0}\}$ and $(a, p) = 1$. Then $ax + py = 1$ for some $x, y \in \mathbb{Z}$. Thus, $ax = 1 - py \equiv 1$ (mod P). There are three cases in the following.

   (a) if $x \in \mathbb{Z}_p - \{\overline{0}\}$, we are done!.

   (b) if $x \notin \mathbb{Z}_p - \{\overline{0}\}$, then $x = pk + x'$ where $x' \in \mathbb{Z}_p - \{\overline{0}\}$.
   $ax + py = 1 \Rightarrow a(pk + x') + py = 1 \Rightarrow apk + ax' + py \equiv 1$ (mod P)

   (c) if $x = 0$, then $py = 1$. $\rightarrow\leftarrow$

3. Let A, B, C be any three eight-letter words, denoted by A $= a_1a_2a_3a_4a_5a_6a_7a_8$, B $= b_1b_2b_3b_4b_5b_6b_7b_8$, C $= c_1c_2c_3c_4c_5c_6c_7c_8$.

$$\begin{aligned}(A \oplus B) \oplus C &= a_1a_2a_3a_4a_5b_6b_7b_8 \oplus c_1c_2c_3c_4c_5c_6c_7c_8 \\ &= a_1a_2a_3a_4a_5c_6c_7c_8\end{aligned}$$

$$\begin{aligned}A \oplus (B \oplus C) &= a_1a_2a_3a_4a_5a_6a_7a_8 \oplus b_1b_2b_3b_4b_5c_6c_7c_8 \\ &= a_1a_2a_3a_4a_5c_6c_7c_8\end{aligned}$$

   This implies $(A \oplus B) \oplus C = A \oplus (B \oplus C)$. Thus it is a semigroup. Clearly, it isn't monoid.

4. Let A, B, C be any three eight-letter words, denoted by A $= a_1a_2a_3a_4a_5a_6a_7a_8$, B $= b_1b_2b_3b_4b_5b_6b_7b_8$, C $= c_1c_2c_3c_4c_5c_6c_7c_8$.

$$\begin{aligned}(A \oplus B) \oplus C &= a_5a_6a_7a_8b_1b_2b_3b_4 \oplus c_1c_2c_3c_4c_5c_6c_7c_8 \\ &= b_1b_2b_3b_4c_1c_2c_3c_4\end{aligned}$$

$$\begin{aligned}A \oplus (B \oplus C) &= a_1a_2a_3a_4a_5a_6a_7a_8 \oplus b_5b_6b_7b_8c_1c_2c_3c_4 \\ &= a_5a_6a_7a_8b_5b_6b_7b_8\end{aligned}$$

   This implies $(A \oplus B) \oplus C \neq A \oplus (B \oplus C)$. Thus it isn't a semigroup.