1. (a) We need to find A_i with | A_i | = P^i for 1 ≤i≤ n
P | |G| = P^n, ∃ a ∈ G s.t. <a> ◁ G with | < a_i > | = P by Cauchy Thm
Hence A_1 = P
Consider G/<a_i> = G/A_1,
then P | | G/A_1 | = P^{n-1}
Then ∃ a_2 ∈ G s.t. < a_2 > ◁ G/A_1 with |<a_2>| = P
Use induction or reference to previous homework.

(b) Pf.1
By corollary 5.4, | Z_i(G) | 1 for all i
Since Z(G) ◁ G
|Z(G)| | |G| ⇒ |Z(G)| = P^i, i<n (if i=n, done)
Inductively, G=Z_m(G) for some m

Pf.2
| H |=P^k ⇒ | Z(H)| ≥ P
| Z_1(G) | = | Z(G)| ≥ P
| Z_2(G) | = | Z_2(G)/Z_1(G) | × Z_1(G) ≥ p × p = p^2
By induction, | Z_i(G) | ≥ p^i
Hence Z_n(G) = G

(c) G = H × K
Claim:Z_i(G) = Z_i(H) × Z_k(K)
And Π_H be the canonocal epimorphism H ⟶ H/Z_i(H)
Π_K be the canonocal epimorphism K ⟶ K/Z_i(K)
⇒ Π = Π_H × Π_K
ϕ: G → G/Z_i(G)
$$G = H \times K \overrightarrow{\Pi} H/Z_i(H) \times K/Z_i(K) \overrightarrow{\Phi} \frac{H \times K}{Z_i(H) \times Z_i(K)} = \frac{G}{Z_i(G)}$$
$Z_{i+1}(G) = \Phi^{-1}(Z(G/Z_i(G)))$
= $\Pi^{-1}\Phi^{-1}$[Z(G/($Z_i(G)$))]
= Π^{-1}[Z(H/$Z_i(H)$× K/$Z_i(K)$)]
= Π^{-1}[Z(H/$Z_i(H)$× Z(K/$Z_i(K)$))]
= Π_H^{-1}(Z(H/$Z_i(H)$)) × Π_H^{-1}[Z(K/$Z_i(K)$)]
= $Z_{i+1}(H)$ × $Z_{i+1}(K)$
∵ H, K are nilpotent
∃ n ∈ ℕ s.t. $Z_n(H)$ = H, $Z_n(K)$ = K
$Z_n(G)$ = $Z_n(H)$ × $Z_n(K)$ = H × K = G

(d) Ascending central series
<e> = $Z_0(G)$ < $Z_1(G)$ < $Z_2(G)$ < ... < $Z_n(G)$ = G
∃ k s.t. $Z_k(G)$ < H and $Z_{k+1}(G) \not<$ G
Choose a ∈ $Z_{k+1}(G)$ with a not in H
∀ h ∈ H, $Z_k(ah) = (Z_k a)(Z_k h) = (Z_k h)(Z_k a) = Z_k(ha)$ in G/$Z_k(G)$
⇒ ah = h'ha, for some h' ∈ $Z_k(G)$ < H
⇒ a ∈ $N_G(H)$
⇒ H ⩽ $N_G(H)$

(e) G = P_1 × P_2 × ... P_k
Want to show:
1.Normal

2.$P_i \cap P_1P_2...P_{i-1}P_{i+1}...P_k$ = {e}
3.G=$P_1P_2...P_k$

1.Suppose P is a Sylow P-subgroup of G for some prime P
(i)If p = G clear.
(ii)IF P $\lneqq$ G proper subgroup
By (d) P is a proper sungroup of $N_G(P)$
P normal: gPg^{-1} = P ∀ P ∈ G
⇔ $N_G(P)$ = { g∈G | gPg^{-1}=P } = G
Claim: $N_G(N_G(P)) = N_G(P)$
⊇ Clear
⊆ ∵ P ◁ $N_G(P)$ By 2nd Sylow Thm
P is the only Sylow P-subgroup of $N_G(P)$
x ∈ $N_G(N_G(P))$ ($N_G(H)$ = H, h=G)
⇒ x$N_G(P)x^{-1}$ = $N_G(P)$
⇒ xPx^{-1} < $N_G(P)$
⇒ xPx^{-1} = P
⇒ x ∈ $N_G(P)$
Hence $N_G(N_G(P)) = N_G(P)$
⇒ $N_G(P)$ = G
⇒ P ◁ G

2.Let |G| = $p_1^{n_1}p_2^{n}...p_k^{n_k}$ where p_i are dictinct prime and $n_i > 0$
Let P_i be the Sylow P_i-subgroup of G
⇒ P_i ◁ G
P_i, P_j normal, xy = yx ∀ x ∈ P_i y ∈ P_j
$P_i \cap P_j$ = {e} ∀ i ≠ j
∀ i, $P_1P_2...P_{i-1}P_{i+1}...P_k$ is subgroup of order $p_1^{n_1}p_2^{n}...p_k^{n_k}$
Since P_i is prime ∀ i ⇒ (P_i, P_j) = 1 ∀ i ≠ j
⇒ $P_i \cap P_1P_2...P_{i-1}P_{i+1}...P_k$ = {e}

3. P ◁ G = | $P_1P_2...P_k$ | = | $P_1 \times P_2 \times ... \ P_k$ |
⇒ G = $P_1 \times P_2 \times ... \ P_k$