Solution for Homework 3 part 2, problem 6 to 7

6.Two element $a, b \in G$ are *conjugate* if there exists $c \in G$ such that $a = c^{-1}bc$.The conjugate is an equivalent relation on G and hence G is partitioned into *conjugate classes.*

(a) Find the number of conjugates of (1,2)(3,4) in S_n , n≥4.

(b) Find the form of all elements commuting with (1,2)(3,4) in S_n , n≥4.

(c) Determine number the conjugate classes of S_6 .

Solution:

(a) $g($ 1 2 $\cdots$ t $)g^{-1}$=($g(1)$ $g(2)$ $\cdots$ $g(t)$) , $\forall g \in S_n$

a(1 2)(3 4)a^{-1}=($a(1)$ $a(2)$)($a(3)$ $a(4)$)

$a(1)$, $a(2)$, $a(3)$, $a(4) \in \{1, 2, \cdots, n\}$, and they are distinct.

⇒there are $n(n-1)(n-2)(n-3)$ ways for choosing $a(1)$,$a(2)$,$a(3)$,$a(4)$.

But $a(1)$ $a(2)$ can be exchange , also $a(3)$ $a(4)$ can.

Similarly , ($a(1)$ $a(2)$) , ($a(3)$ $a(4)$) can be exchange.

Hence , there are $\frac{n(n-1)(n-2)(n-3)}{2*2*2}$ conjugates.

(b)Let S={e,(1 2),(3 4),(1 2)(3 4),(1 3)(2 4),(1 4)(2 3),(1 3 2 4),(1 4 2 3)}

Every element in S can commute with (1 2)(3 4).

Let $S' = \{\tau \in S_n \mid \tau$ is a permutation leaving all 1,2,3,4}

Fix $\tau \in S'$, $a \in S$, $a\tau$ commutes with (1 2)(3 4)

σ(1 2)(3 4)=(1 2)(3 4)σ

⇒ σ(1 2)(3 4)σ^{-1}=(1 2)(3 4)

⇒ ($\sigma(1)$ $\sigma(2)$)($\sigma(3)$ $\sigma(4)$)=((1 2)(3 4)

⇒ $\sigma \in S\tau$

The answer is $\{a\tau | a \in S\}$.

(c)$g($ 1 2 $\cdots$ t $)g^{-1}$=($g(1)$ $g(2)$ $\cdots$ $g(t)$)

⇒ The length of (1 2 $\cdots$ t) is the same with ($g(1)$ $g(2)$ $\cdots$ $g(t)$).

$a = c^{-1}bc$ ⇒ The length of a is the same with b.

The conjugates of S_6 :

(1) ε (2) 2-cycle (3) 3-cycle (4) 4-cycle (5) 5-cycle (6) 6-cycle

(7) product of two 2-cycle (8) product of three 2-cycle

(9) product of 2-cycle and 3-cycle (10) product of 2-cycle and 4-cycle

(11) product of two 3-cycle

There are 11 conjugate classes of S_6.

Teacher:

6+0 = 5+1 = 4+2 = 4+1+1

= 3+3 = 3+2+1 = 3+1+1+1

= 2+2+2 = 2+2+1+1 = 2+1+1+1+1

= 1+1+1+1+1+1

⇒ 11 ways

7.Let the column vector $u = (u_1, u_2, u_3)^t$ represent a coloring configuration of the path $P_3 = \{1-2-3\}$ described in Homework 1 , where $u_i \in Z_2$; u_i=0 iff the vertex i is colored in black (off).

(a) Interpret each lit-only move associated with a vertex i "faithfully" to a 3×3 matrix S_i with entries over Z_2 such that S_i sends a configuration u to $S_i u$.

(b)Interpret each dual lit-only move associated with a vertex i "faithfully" to a 3×3 matrix S_i^* with entries over Z_2 such that S_i^* sends a configuration u to $S_i^* u$.What is the relation between S_i and S_i^*.

(c) Interpret each dual lit-only plus move associated with a vertex i "faithfully" to a 3×3 matrix M_i with entries over Z_2 such that M_i sends a configuration u to $M_i u$.

(d) Let $\mathbf{L} =< S_1, S_2, S_3 >$. Show that the center Z($\mathbf{L}$) of $\mathbf{L}$ is trivial. (Hint.Compute $S_i S$ and SS_i if there exists $S \in Z(\mathbf{L})$ with $S_{ij} = 1$.)

Solution:

(a) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ x+y \\ z \end{bmatrix} \Rightarrow S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x+y \\ y \\ y+z \end{bmatrix} \Rightarrow S_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ y+z \\ z \end{bmatrix} \Rightarrow S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x+y \\ y \\ z \end{bmatrix} \Rightarrow S_1^* = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ x+y+z \\ z \end{bmatrix} \Rightarrow S_2^* = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ y+z \end{bmatrix} \Rightarrow S_3^* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

S_i^* is the transfer of S_i

(c) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \to \begin{bmatrix} x+x \\ x+y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ x+y \\ z \end{bmatrix} \Rightarrow M_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x+y \\ y+y \\ y+z \end{bmatrix} = \begin{bmatrix} x+y \\ 0 \\ y+z \end{bmatrix} \Rightarrow M_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y+z \\ z+z \end{bmatrix} = \begin{bmatrix} x \\ y+z \\ 0 \end{bmatrix} \Rightarrow M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

(d)Suppose $S \in Z(L), S = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_1+a_4 & a_2+a_5 & a_3+a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} = S_1S = SS_1 = \begin{bmatrix} a_1+a_2 & a_2 & a_3 \\ a_4+a_5 & a_5 & a_6 \\ a_7+a_8 & a_8 & a_9 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1 = a_1 + a_2 \Rightarrow a_2 = 0 \\ a_1 + a_4 = a_4 + a_5 \Rightarrow a_1 = a_5 \\ a_3 + a_6 = a_6 \Rightarrow a_3 = 0 \\ a_7 = a_7 + a_8 \Rightarrow a_8 = 0 \end{cases}$$

$$\begin{bmatrix} a_1+a_4 & a_2+a_5 & a_3+a_6 \\ a_4 & a_5 & a_6 \\ a_4+a_7 & a_5+a_8 & a_6+a_9 \end{bmatrix} = S_2S = SS_2 = \begin{bmatrix} a_1 & a_1+a_2+a_3 & a_3 \\ a_4 & a_4+a_5+a_6 & a_6 \\ a_7 & a_7+a_8+a_9 & a_9 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a_1 + a_4 = a_1 \Rightarrow a_4 = 0 \\ a_3 + a_6 = a_3 \Rightarrow a_6 = 0 \\ a_5 + a_8 = a_7 + a_8 + a_9 \Rightarrow a_5 = a_7 + a_9 \end{cases}$$

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4+a_7 & a_5+a_8 & a_6+a_9 \\ a_7 & a_8 & a_9 \end{bmatrix} = S_3S = SS_3 = \begin{bmatrix} a_1 & a_2 & a_2+a_3 \\ a_4 & a_5 & a_5+a_6 \\ a_7 & a_8 & a_8+a_9 \end{bmatrix}$$

$\Rightarrow a_4 + a_7 = a_4 \Rightarrow a_7 = 0 \Rightarrow a_5 = a_9$

$S = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_5 & 0 \\ 0 & 0 & a_9 \end{bmatrix}, a_1 = a_5 = a_9 = 1 or 0$

if $a_1 = a_5 = a_9 = 0 \Rightarrow S = [0]$ has no inverse.

Hence,$a_1 = a_5 = a_9 = 1 \Rightarrow S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$