1. Define $\phi : G \rightarrow$ G $\times$ N by g=hn $\rightarrow$ (h,n)

   (a) check:1-1
   If ϕ(hn)=ϕ(h'n')
   $\Rightarrow$ (h,n)=(h',n') $\Rightarrow$ h=h', n=n' $\Rightarrow$ hn=h'n'

   (b) onto
   $\forall$ (h,n) $\in$ PN
   $\exists$ g=hn s.t. ϕ (hn) = (h,n)

   (c) well-defined
   Let hn=h'n' $\Rightarrow$ ϕ(hn) = (h,n) and ϕ(h'n') = (h',n')
   $\Rightarrow$ (h,n) = (h',n') $\Rightarrow$ h=h', n=n'

   (d) homo
   ϕ(hnh'n')=ϕ(hh'nn') = (hh',nn') = (h,n) $\times$ (h',n') = ϕ(hn)ϕ(h'n')

2. Pf.1
   Define ϕ:G1 X G2 X G3 $\rightarrow$ G by ϕ(g1,g2,g3) $\rightarrow$ g1g2g3

   (a) 1-1:
   Claim: kerϕ=e
   Let (g1,g2,g3) $\in$ kerϕ $\forall$ g1$\in$G1, g2$\in$G2, g3$\in$G3
   ϕ(g1,g2,g3)=e=g1g2g3 (*)
   $\Rightarrow$ g_3=$(g1g2)^{-1}$
   $\therefore$ g_3 $\in$G1G2 and g_3 $\in$G3
   $\therefore$ g_3 $\in$G3$\wedge$G2$\wedge$G3=e
   So in (*) $\Rightarrow$ e=g1(g2e) $\Rightarrow$ $g1 = (g2e)^{-1}$ $\Rightarrow$ g_1 $\in$ G2G3 and g_1 $\in$G1 $\Rightarrow$ g_1=e
   $\therefore$ e=eg_2e =g_2
   $\therefore$ kerϕ = (e,e,e,)
   $\therefore$ ϕ is 1-1

   (b) onto
   $\forall$ g $\in$G=G1G2G3 $\exists$g1$\in$ g2$\in$G2 g3$\in$G3 s.t. g=g1g2g3=ϕ(g1,g2,g3)
   $\therefore$ $\exists$(g1,g2,g3) $\in$ G1 $\times$ g2 $\times$ g3

   (c) homo
   Let(g1g2g3)(g1'g2'g3') $\in$ G1 $\times$ G2 $\times$ g3
   $\Rightarrow$ ϕ((g1g2g3)(g1'g2'g3'))=ϕ(g1g1',g2g2',g3g3')= (g1g1')(g2g2')(g3g3')
   The method is just like question1, checking by self...

   Pf.2
   Define g:G1 $\times$ G2$\rightarrow$G1G2 by g(g1,g2)=g1g2
   By question.1 g is homo
   Define f:G1G2 $\times$ G3 $\rightarrow$G1G2G3 by f(a1,a2,a3)=a1a2a3.

3. S3=e,(1,2),(1,3),(2,3),(1,2,3),(1,3,2)
   Proper sungroups:e,e,(1,2),e,(1,3),e,(2,3),e,(1,2,3),(1,3,2)
   | S3 |=6
   ex:
   e,(1,2)Xe,(1,2,3),(1,3,2)
   order (e,e)=1, (e,(1,2,3))=3, (e,(1,3,2))=3, ((1,2),e)=2 ((1,2),(1,2,3))=6, ((1,2),(1,3,2))=6.

$\{ e \} \neq H < S3 \Rightarrow | H | = 2$ or $3 \Rightarrow H \cong \mathbb{Z}_2$ or $\mathbb{Z}_3 \Rightarrow \mathbb{Z}_n \times \mathbb{Z}_m$ is abelian
$D_3 = S_3$ is not abelian

4. Suppose yes
   $\mathbb{Z}_{p^n} = H_1 \times H_2 \times ... \times H_s$
   $|Hi| \mid p^n$
   $\because \mathbb{Z}_{p^n}$ is cyclic $\Rightarrow$ Hi is cyclic
   $\therefore \mathbb{Z}_{p^n} \cong \mathbb{Z}_{p^{i1}} \times \mathbb{Z}_{p^{i2}} \times ... \times \mathbb{Z}_{p^n}\ p^{is}$ where $i_1 \leq i_2 \leq ... \leq i_s <$ n

5. $\mathbb{Z} = <m>+<n>$ m,n $\in \mathbb{Z}$
   $<n> \wedge <n> = \{e\} = \{0\}$
   mn $\in <m> \wedge <n>$