HOMEWORK 4

Q6. Is it possible for a cyclic group to be a direct product of its two proper subgroups?

Sol. Yes. For example, let $G = \{e, a, \ldots, a^5\}$ and $G = \langle a^2 \rangle \times \langle a^3 \rangle$.

Q7. If G is a group and $N \lhd G$, show that if $a \in G$ has finite order $|a|$, then Na in G/N has finite order m, where m divides $|a|$.

Proof. Suppose $|a| = n$, i.e., $a^n = e$ in G. $(Na)^n = \underbrace{(Na)\cdots(Na)}_{n} = Na^n = Ne = N$ in G/N. Thus Na has finite order m, $m|n$. □

Q8. Let G be a finite group, α an automorphism of G, and set

$$I = \{g \in G \mid \alpha(g) = g^{-1}\}.$$

(a) Suppose $|I| > \frac{3}{4}|G|$. Show that G is abelian. (Hint. $I \cap h^{-1}I \subseteq C_G(h)$ for $h \in I$.)

Proof.

1. $C_G(h) = \{a \in G \mid ha = ah\} = \{a \in G \mid h = a^{-1}ha\}$.
   Thus $C_G(h)$ is a subgroup of G.
2. Now we show that $I \cap h^{-1}I \subseteq C_G(h)$ for $h \in I$. Suppose $x \in I \cap h^{-1}I$.
   Let $x = h^{-1}c$ for some $c \in I$. $x^{-1} = \alpha(x) = \alpha(h^{-1}hx) = \alpha(h^{-1})\alpha(hx) = hx^{-1}h^{-1}$.
   $\therefore x = hxh^{-1}$, $xh = hx \Rightarrow x \in C_G(h) \Rightarrow I \cap h^{-1}I \subseteq C_G(h)$
3. $|I| = |h^{-1}I| > \frac{3}{4}|G|$
   $\because I \cap h^{-1}I \subseteq G$
   $\therefore |I \cap h^{-1}I| \leq |G|$
   If $I, h^{-1}I$ disjoint, $\rightarrow\leftarrow$.
   Therefore, $|I \cap h^{-1}I| = |I| + |h^{-1}I| - |I \cup h^{-1}I| > (\frac{3}{4} + \frac{3}{4} - 1)|G| = \frac{1}{2}|G|$.
   By 2, $I \cap h^{-1}I \subseteq C_G(h)$
   $\therefore |C_G(h)| > \frac{1}{2}|G|$.
4. By Largrange's Theorem, $C_G(h) = G$.
   This means $Z(G) := \{x \in G \mid xy = yx, \forall y \in G\}$, $Z(G) < G$.
   $\because \forall h \in I, I \subseteq Z(G)$
   $\therefore |Z(G)| > |I| > \frac{3}{4}|G| \Rightarrow Z(G) = G$
   $\Rightarrow G$ is abelian.

□

(b) Suppose $|I| = \frac{3}{4}|G|$. Show that G has an abelian subgroup of index 2. (Hint. Consider $C_G(h)$ for $h \in I - Z(G)$.)

Proof. Use (a). $I \cap h^{-1}I \subseteq C_G(h)$ for all $h \in I$.
We know $|I \cap h^{-1}I| = |I| + |h^{-1}I| - |I \cup h^{-1}I| \geq (\frac{3}{4} + \frac{3}{4} - 1)|G| = \frac{1}{2}|G|$.
$\because C_G(h)$ is a group, by Largrange's Theorem, $|C_G(h)| = \frac{1}{2}|G|$ or $|G|$.

1. If $|C_G(h)| = |G|$ for all $h \in I$, then $I < Z(G)$. Hence $Z(G) = G$, and note that I is a group, $(a, b \in I, \alpha(ab^{-1}) = \alpha(a)\alpha(b^{-1}) = a^{-1}b = (b^{-1}a)^{-1} = (ab^{-1})^{-1})$ a contradiction.

2. Assume $|C_G(h)|$ is abelian. Pick $a, b \in C_G(h) = I \cap h^{-1}I$.

   Note $\alpha(aba^{-1}b^{-1}) = \alpha(ab)\alpha(a^{-1})\alpha(b^{-1}) = (ab)^{-1}(a^{-1})^{-1}(b^{-1})^{-1} = b^{-1}a^{-1}ab = e$.

   Hence $ab = ba$. Then $C_G(h)$ is what we want.

$\square$