進階代數(下) 第七次作業

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2009 年四月六日

1. Suppose $G = H \times K$
   (a) Let N be a normal subgroup of H. Show that N is normal in G.
   (b) Suppose that G satisfies the $ACCN$. Show that H satisfies $ACCN$.
   (c) Suppose that G satisfies the $DCCN$. Show that H satisfies $DCCN$.

pf :

(a) We want to show $nk = kn \quad \forall n \in N \lhd H \quad \forall k \in K$

$\because \; nkn^{-1}k^{-1} \in N \lhd H$ and $nkn^{-1}k^{-1} \in K$ and $H \cap K = \{e\}$

$\therefore nkn^{-1}h^{-1} = e$ ➔ $nk(kn)^{-1} = e$ ➔ $nk = kn$ --(*)

$$gng^{-1} = hkn(hk)^{-1} = hknk^{-1}h^{-1} \xrightarrow{by(*)} hnkk^{-1}h^{-1} = hnh^{-1} \in N$$

(b) Assume H is not $ACCN$.

$\exists$ a chain $H_1 < H_2 < .....$ of normal subgroup of H.

there is no integer n such that $H_i = H_n \quad \forall i > n$

By (a) $\because H_i \lhd H \quad \therefore H_i \lhd G$ ➔ G is not $ACCN$.

(c) By (a) then every decending chain of normal subgroup of H is a chain of normal subgroup of G. ➔ H has $DCCN$.

2. Let Q_8 be the multiplication group generated by the complex matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Q_8 is called the *quaternion group.*

(a) Show $|Q_8| = 8$.

(b) *Consider the set $G = \{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication given by $i^2 = j^2 = k^2 = -1; ij = k = -ji; jk = i = -kj; ki = j = -ik$, and the usual rules for multiplying by ± 1. Show that G is a group isomorphic to the quaternion group Q_8*:*

(c) What is the center $Z(Q_8)$ of the quaternion group Q_8?

(d) Show that $Q_8 / Z(Q_8)$ is abelian.

(e) Is Q_8 isomorphic to $(Z(Q_8) \times Q_8 / Z(Q_8))$.

pf :

(a) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I = B^2$

$AB = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$BA = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$

$ABA = B \qquad BAB = A$

$Q_8 = \{I, A, B, AB, -I, -A, -B, BA\}$

➔ $|Q_8| = 8$

(b) Let $i = A, j = B, k = AB, 1 = I$

then $i^2 = j^2 = k^2 = -1; ij = k = -ji; jk = i = -kj; ki = j = -ik$.

(c) $G \cong Q_8 \quad \begin{cases} ij = -ji \\ jk = -kj \end{cases} \Rightarrow Z(G) = \{\pm 1\} \Rightarrow Z(Q_8) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

(d) $Q_8 / Z(Q_8) = \{I, A, B, AB\}$

We want to show $AB = BA$

$AZ(Q_8)BZ(Q_8) = ABZ(Q_8) = -BAZ(Q_8) = -IBAZ(Q_8) = BAZ(Q_8)$

(e) Q_8 is not abelian by $AB = -BA$

$(Z(Q_8) \times Q_8 / Z(Q_8))$ is abelian by (d)

So Q_8 is not isomorphic to $(Z(Q_8) \times Q_8 / Z(Q_8))$