進階代數(下) 第七次作業解答

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3. (a) No. Given a counter-example.

$$\begin{aligned} G &= D_3 = \{e, a, a^2, b, ba, ba^2\} \\ N &= \{e, a, a^2\} \lhd G \\ H &= \{e, b\} < G \end{aligned}$$

but N, H are abelian, $G = D_3$ is not abelian.

(b) For $n_1, n_2 \in N$,
Homo:

$$\begin{aligned} \phi_h(n_1 \cdot n_2) &= h(n_1 n_2)h^{-1} \\ &= hn_1h^{-1}hn_2h^{-1} \\ &= \phi_h(n_1) \cdot \phi_h(n_2) \end{aligned}$$

One-to-one:

$$\begin{aligned} \phi_h(n_1) = \phi_h(n_2), \text{ for } n_1, n_2 \in N \\ hn_1h^{-1} = hn_2h^{-1} \Rightarrow n_1 = n_2 \end{aligned}$$

Onto: For any $n \in N$

$$\begin{aligned} \because N \lhd G, \therefore h^{-1}nh \in N \\ \phi_h(n^{-1}nh) = hh^{-1}nhh^{-1} = n \end{aligned}$$

(c) Check $\phi(ab) = \phi(a) \circ \phi(b)$ for $a, b \in H$.
For $n \in N$,

$$\begin{aligned} \phi(ab)(n) &= \phi_{ab}(n) \\ &= abn(ab)^{-1} \\ &= a(bnb^{-1})a^{-1} \\ &= a(\phi_b(n))a^{-1} \\ &= \phi_a(\phi_b(n)) \\ &= (\phi(a) \circ \phi(b))(n) \end{aligned}$$

(d) For $n, n' \in N$, $h, h' \in H$,

$$\begin{aligned} nhn'h' &= n\underline{hn'h^{-1}}hh' \\ &= n\phi_h(n')hh' \end{aligned}$$

(e) Let

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

Then let $N =< \sigma >$, $H = \{I, \tau\}$.
$N \lhd D_8$, $H < D_8$, $N \cap H = \{I\}$ and $D_8 = HN$.

2 (a) Yes, it's (e_N, e_H). For $(n, h) \in N \times H$,

$$(n,h) \cdot (e_N, e_H) = (n\underline{\phi(h)e_N}, he_H) = (n\underline{e_N}, h) = (n,h).$$

Since $\phi(h)$ is $Aut(N)$, $\phi(h)$ is homomorphism. Thus $\phi(h)(e_N) = e_N$.

$$(e_N, e_H) \cdot (n,h) = (\underline{e_N \phi(e_H)}n, e_H h) = (\underline{e_N}n, h) = (n,h).$$

Since $\phi(e_H) = \phi(e_H \cdot e_H) = \phi(e_H) \circ \phi(e_H)$, $\phi(e_H)(n) = n$ for $n \in N$.

(b) $(n,h) \cdot (n', h') = (n\phi(h)(n'), hh') = (e_N, e_H)$.

* Since H is a group, there exists $h^{-1} = h'$ such that $hh' = e_H$.
* Consider $\phi(h)(n') = n^{-1}$.
  Since $\phi : H \to Aut(N)$, $\phi(h) \in Aut(N)$.
  Thus these exists $n' \in N$ such that $\phi(h)(n') = n^{-1}$.
  So $n' = (\phi(h))^{-1}(n^{-1})$.

Therefore, $(n,h)^{-1} = ((\phi(h))^{-1}(n^{-1}), h^{-1})$.

(c) By (a), identity. By (b), inverse. Thus we must show closed and associative.

Closed: Take $(n,h), (n', h') \in N \times H$.

$(n,h) \cdot (n', h') = (n\phi(h)(n'), hh') \in N \times H$ since $\phi(h) : N \longrightarrow N$.

Associative: Take $(n_1, h_1), (n_2, h_2), (n_3, h_3) \in N \times H$.

$$\begin{aligned} ((n_1,h_1)(n_2,h_2))(n_3,h_3) &= (n_1\phi(h_1)(n_2), h_1h_2)(n_3,h_3) \\ &= (n_1\phi(h_1)(n_2)\phi(h_1h_2)(n_3), h_1h_2h_3) \\ &= (n_1\phi(h_1)(n_2)(\phi(h_1)\phi(h_2))(n_3), h_1h_2h_3) \end{aligned}$$

$$\begin{aligned} (n_1,h_1)((n_2,h_2)(n_3,h_3)) &= (n_1,h_1)(n_2\phi(h_2)(n_3), h_2h_3) \\ &= (n_1\phi(h_1)(n_2\phi(h_2)(n_3)), h_1h_2h_3) \\ &= (n_1\phi(h_1)(n_2)\phi(h_1)(\phi(h_2)(n_3)), h_1h_2h_3) \end{aligned}$$