

(a) $G = Z_2 \times Z_4 \times Z_9$

Take $H = \{e\} \times Z_4 \times Z_9$, $K = Z_2 \times \{e\} \times \{e\}$

Then we have to check G is the inner direct product of H , K .

First, it is clearly that $H \triangleleft G$, $K \triangleleft G$.

Second, $\forall (a, b, c) \in G$

$\exists (a, b, c) \in H$ and $(a, 0, 0) \in K$ such that $(a, b, c) = (0, b, c) \cdot (a, 0, 0)$
 $\Rightarrow G \subset H \times K$.

Third, $H \cap K = \{e\}$.

Therefore, $G = H \times K$.

(b)

Take $H = \{0\} \times Z_4 \times Z_9$, and $K = \{0\} \times Z_2 \times \{0\}$.

Then there exists $(0, 2, 0) \in H \cap K$.

Hence, $G \neq H \times K$.

(c)

Take $H = Z_2 \times \langle 2 \rangle \times Z_9$.

Suppose $\exists K < G$ s.t $G = H \times K$ where $|G| = 72$ and $|H| = 36$.

Then $|K| = 2$.

Let $h = \langle (0, 1, 0) \rangle < G$.

Then $|h| = 4$.

But $H \times K$ no subgroup of order 4.

So for any subgroup K of G , $G \neq H \times K$.

(d)

(e)

First, since $|G| < \infty$, G satisfies ACCN, DCCN.

So, G is the direct product of finite indecomposable normal subgroups.

Second, By Krull-schmit Theorem, $G = G_1 \times G_2 \times \cdots \times G_k$, where G_i is indecomposable.

Moreover, by (d), G_i is cyclic.

So, $G_i \cong Z_m$ for some m .

By the way, if $m = lk$ for $(l, k) = 1$, then $Z_m \cong Z_l \times Z_k$ is decomposable.

Hence, $m = p_i^{s_i}$.

Thus $G = G_1 \times G_2 \times \cdots \times G_k$ (inner product) $\cong G_1 \times G_2 \times \cdots \times G_k$ (direct product) $\cong Z_{p_1^{s_1}} \times Z_{p_2^{s_2}} \times \cdots \times Z_{p_k^{s_k}}$

(f)

False.

Take $G = G_1 \times G_2 = H_1 \times H_2$, where $G_1 \cong G_2 \cong Z_4$, $H_1 \cong Z_2$, $H_2 \cong Z_8$.

The reason is that $Z_4 \times Z_4 \neq Z_2 \times Z_8$.

Since $\forall x \in Z_4 \times Z_4$, the order of x is at most 4.

But when we take $(0, 1) \in Z_2 \times Z_8$, the order of $(0,1)$ is 8.

Thus, $Z_2 \times Z_4 \neq Z_2 \times Z_8$.

(g)

For $1400 = 2^3 \times 5^2 \times 7$, we have $3 \times 2 \times 1 = 6$ non-isomorphic abelian groups of order 1400.