Tridiagonal pairs in Lie theory, quantum groups, and orthogonal polynomials

by Paul Terwilliger (University of Wisconsin)

In this talk we consider a linear algebraic object called a tridiagonal pair, and the role it plays in Lie theory, quantum groups, and orthogonal polynomials. We aim at a general mathematical audience; no prior experience with the above topics is assumed. Our main results are joint work with Tatsuro Ito.

The concept of a tridiagonal pair is best explained by starting with a special case known as a Leonard pair. Let $\mathbb{F}$ denote a field, and let V denote a vector space over $\mathbb{F}$ with finite positive dimension. By a *Leonard pair* on V we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following two conditions:

1. There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is irreducible tridiagonal;

2. There exists a basis for V with respect to which the matrix representing A^* is diagonal and the matrix representing A is irreducible tridiagonal.

We recall that a tridiagonal matrix is irreducible whenever each entry on the superdiagonal is nonzero and each entry on the subdiagonal is nonzero. The name "Leonard pair" is motivated by a connection to a 1982 theorem of the combinatorialist Doug Leonard involving the q-Racah and related polynomials of the Askey scheme. Leonard's theorem was heavily influenced by the work of Eiichi Bannai and Tatsuro Ito on P- and Q- polynomial association schemes, and the work of Richard Askey on orthogonal polynomials. These works in turn were influenced by the work of Philippe Delsarte on coding theory, dating from around 1973.

The central result about Leonard pairs is that they are in bijection with the orthogonal polynomials that make up the terminating branch of the Askey scheme. This branch consists of the q-Racah, q-Hahn, dual q-Hahn, q-Krawtchouk, dual q-Krawtchouk, quantum q-Krawtchouk, affine q-Krawtchouk, Racah, Hahn, dual-Hahn, Krawtchouk, and the Bannai/Ito polynomials. The bijection makes it possible to develop a uniform theory of these polynomials starting from the Leonard pair axiom, and we have done this over the past several years.

Leonard pairs are related to orthogonal polynomials as follows. Given a Leonard pair A, A^* fix a basis for the underlying vector space that diagonalizes A and makes A^* irreducible tridiagonal. Let that tridiagonal matrix represent the three-term recurrence for a graded sequence of polynomials. The resulting polynomials are from the terminating branch of the Askey scheme.

A tridiagonal pair is a generalization of a Leonard pair and defined as follows. Let V denote a vector space over $\mathbb{F}$ with finite positive dimension. A *tridiagonal pair on* V is an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy the following four conditions:

(i) Each of A, A^* is diagonalizable on V;

(ii) There exists an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A such that

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad (0 \le i \le d),$$

where $V_{-1} = 0, V_{d+1} = 0$;

(iii) There exists an ordering $\{V_i^*\}_{i=0}^\delta$ of the eigenspaces of A^* such that

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad (0 \le i \le \delta),$$

where $V_{-1}^* = 0, V_{\delta+1}^* = 0$;

(iv) There is no subspace $W \subseteq V$ such that $AW \subseteq W$, $A^*W \subseteq W$, $W \neq 0$, $W \neq V$.

It turns out that $d = \delta$ and this common value is called the diameter of the pair. A Leonard pair is the same thing as a tridiagonal pair for which the eigenspaces V_i and V_i^* all have dimension 1.

Tridiagonal pairs arise naturally in the theory of P- and Q- polynomial association schemes, in connection with irreducible modules for the subconstituent algebra. They also appear in recent work of Pascal Baseilhac on the Ising model and related structures in statistical mechanics.

It is an open problem to classify the tridiagonal pairs, but we are making progress. One of the things we recently discovered is a connection between tridiagonal pairs on one hand, and Lie theory and quantum groups on the other; this connection is the main topic of the talk. In order to explain our results we make some comments on Lie theory. Recall that the Lie algebra $\mathfrak{sl}_2$ has a basis e, f, h that satisfies $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. It turns out that $\mathfrak{sl}_2$ has another basis x, y, z such that $[x, y] = 2x+2y$, $[y, z] = 2y+2z$, $[z, x] = 2z + 2x$. With this in mind we generalize $\mathfrak{sl}_2$ as follows.

Definition The *tetrahedron algebra* $\boxtimes$ is the Lie algebra over $\mathbb{F}$ that has generators

$$\{x_{ij} \mid i, j \in \mathbb{I}, i \neq j\} \qquad\qquad \mathbb{I} = \{0, 1, 2, 3\}$$

and the following relations:

(i) For distinct $i, j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0.$$

(ii) For mutually distinct $i, j, k \in \mathbb{I}$,

$$[x_{ij}, x_{jk}] = 2x_{ij} + 2x_{jk}.$$

(iii) For mutually distinct $i, j, k, \ell \in \mathbb{I}$,

$$[x_{ij}, [x_{ij}, [x_{ij}, x_{k\ell}]]] = 4[x_{ij}, x_{k\ell}].$$

It turns out that for mutually distinct $i, j, k \in \mathbb{I}$ the elements x_{ij}, x_{jk}, x_{ki} form a basis for a subalgebra of $\boxtimes$ that is isomorphic to $\mathfrak{sl}_2$. One can also find subalgebras of $\boxtimes$ that are isomorphic to the $\mathfrak{sl}_2$ loop algebra. We have shown that $\boxtimes$ is isomorphic to the three-point $\mathfrak{sl}_2$ loop algebra.

Our object of interest is not $\boxtimes$ itself, but rather a certain q-deformation $\boxtimes_q$ called the q-tetrahedron algebra. We show that $\boxtimes_q$ is related to the quantum group $U_q(\mathfrak{sl}_2)$ in roughly the same way that $\boxtimes$ is related to $\mathfrak{sl}_2$, and $\boxtimes_q$ is related to the $U_q(\mathfrak{sl}_2)$ loop algebra in roughly the same way that $\boxtimes$ is related to the $\mathfrak{sl}_2$ loop algebra.

In our main results we obtain a connection between the finite-dimensional irreducible $\boxtimes_q$-modules and tridiagonal pairs. These main results are summarized in the following two theorems and subsequent remark.

Theorem Let V denote a vector space over $\mathbb{F}$ with finite positive dimension and let A, A^* denote a tridiagonal pair on V that has q-geometric type. Then there exists a unique $\boxtimes_q$-module structure on V such that x_{01}, x_{12} act on V as A, A^* repectively. This module is irreducible and type 1.

Theorem Let V denote a finite-dimensional irreducible $\boxtimes_q$-module of type 1. Then the generators x_{01}, x_{23} act on V as a tridiagonal pair of q-geometric type.

Remark Combining the previous two theorems we get a bijection between the following two sets: (i) the isomorphism classes of finite-dimensional irreducible $\boxtimes_q$-modules that have type 1; (ii) the isomorphism classes of tridiagonal pairs over $\mathbb{F}$ that have q-geometric type.

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