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5. Questions

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   Let $h_\alpha = [e^\alpha, e^{-\alpha}]$. Then $\mathrm{span}\{h_\alpha, e^\alpha, e^{-\alpha}\} \cong sl_2(\mathbb{C})$.

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(They are Combinatorial objects!!!)

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Denote $G^n = G \oplus \cdots \oplus G$ and let $\mathcal{L}^n = \mathcal{L} \oplus \cdots \oplus \mathcal{L}$ be the orthogonal sum of n-copies of L.

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Question 2: For a given D, compute $Aut\,D$. Again it is related to the determination of $Aut\,V_{L_D}$ and the properties of the Monster group.

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Is $\sqrt{k}A_\ell^{24/\ell} \subset \Lambda$ for any positive integer k??