Boson-Fermion Correspondence and Jacobi Triple Product Identity

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Organization

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2. Example of Lie (super)algebras

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6. Computation of the Character and Jacobi Triple Product Identity

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- So representation is just a way of realizing something that may be very abstract as matrices.

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- So the notion of representation of G is the SAME as the notion of a G-module.

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  (2) $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$, for all $X, Y, Z \in L$. Jacobi identity.

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  A Lie superalgebra L is a $\mathbb{Z}_2$-graded space, i.e. L is a direct sum of two vector spaces

  $$L = L_{\bar{0}} \oplus L_{\bar{1}},$$

  equipped with a degree-preserving (i.e. $[L_\epsilon, L_\delta] \subseteq L_{\epsilon+\delta}$, $\epsilon, \delta \in \mathbb{Z}_2$) bilinear map $[\cdot,\cdot] : L \times L \longrightarrow L$ satisfying

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  Above X,Y,Z are all homogeneous elements of L and $x = \epsilon$, if $X \in L_\epsilon$.

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  For example: $\psi^+_{\frac{1}{2}}\cdot\psi^-_{-\frac{1}{2}}|0\rangle = -\psi^-_{-\frac{1}{2}}\cdot\psi^+_{\frac{1}{2}}|0\rangle + \mathbf{1}|0\rangle = |0\rangle.$

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$$\alpha_m := \frac{\partial}{\partial x_m}, \quad m > 0,$$

$$\alpha_0 := \lambda.$$

  Above the formulas give the way the elements α_m acts on a polynomial in $\mathbb{C}[x_1, x_2, \cdots]$, for example

$$\alpha_1 \cdot x_1 = 1; \quad \alpha_{-1} x_1 = x_1^2; \quad \alpha_0 x_1 = \lambda x_1.$$

- Let us construct representations of $\mathcal{H}$:

  Let $\mathbb{C}[x_1, x_2, x_3, \cdots]$ be the polynomial ring in infinitely many variables $\{x_1, x_2, x_3, \cdots\}$.

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  This way we obtain a one-parameter family of $\mathcal{H}$-modules, which we will denote by

$$V_\lambda, \quad \lambda \in \mathbb{C}.$$

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  Let us define the normal ordering $: \cdot\cdot :$ of two ψ operators by

$$:\psi_r^{\pm}\psi_s^{\pm}: = -\psi_s^{\pm}\psi_r^{\pm}, \quad \text{if } s < 0 \text{ and } r > 0,$$

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- Introduce three generating series (z an indeterminate):

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\psi^{\pm}(z) := \sum_{r \in \frac{1}{2}+\mathbb{Z}} \psi_r^{\pm} z^{-r-\frac{1}{2}},
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$$
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Hence if we let α_m act by A_m and $\mathbf{1}$ act by 1, then $\mathcal{F}$ is a representation of $\mathcal{H}$.

- Note that A_m is always a sum of infinitely many operators. Let us write down for example

$$A_0 := \sum_{r\in\frac{1}{2}+\mathbb{Z}} : \psi_r^+\psi_{-r}^- := \alpha_0.$$

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We can write the formula for this vector in $\mathcal{F}$, which we denote by $|\lambda\rangle$:

$$\begin{aligned} |\lambda\rangle &= |0\rangle, && \lambda = 0, \\ |\lambda\rangle &= \psi^+_{-\lambda+\frac{1}{2}} \cdots \psi^+_{-\frac{1}{2}} |0\rangle, && \lambda > 0, \\ |\lambda\rangle &= \psi^-_{\lambda+\frac{1}{2}} \cdots \psi^-_{-\frac{1}{2}} |0\rangle, && \lambda < 0. \end{aligned}$$

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- The following formulas are needed later on:

$$[L_0, \alpha_n] = -n\alpha_n,$$

$$[L_0, \psi_r^{\pm}] = -r\psi_r^{\pm},$$

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  The reason to call the above expression Trace of x^T is because if we think of x^T acting on W_μ as x^μ, then the expression is precisely the trace of x^T on W!

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