On the Enumeration of Parking Functions by Leading Numbers

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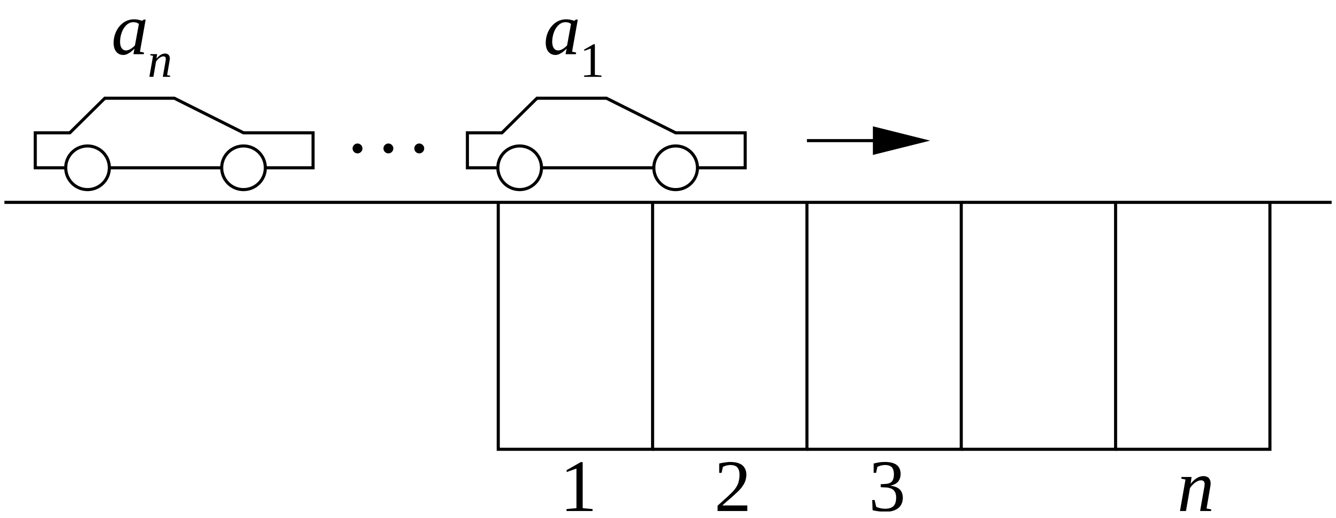
Introduction



For example, $(a_1, a_2, a_3, a_4, a_5) = (2, 1, 3, 5, 2)$ is a parking function, but $(3, 1, 4, 5, 3)$ is not.

Parking functions

A *parking function* of length n is a sequence $\alpha = (a_1, \ldots, a_n)$ of positive integers such that the non-decreasing rearrangement $b_1 \leq \cdots \leq b_n$ of α satisfies $b_i \leq i$.

Konheim and Weiss first derived that the number of parking functions of length n is $(n+1)^{n-1}$ when they dealt with a hashing problem in computer science.

The following numbers are equal to $(n+1)^{(n-1)}$.

1. the number of parking functions of length n,
2. the number of labeled trees on the vertex set $\{0, 1, \ldots, n\}$,
3. the number of regions in the Shi arrangements in $\mathbb{R}^n$,
4. the number of maximal chains in the poset of noncrossing partitions of $\{1, \ldots, n+1\}$,
5. the number of ways to decompose an $(n+1)$-cycle $\sigma \in S_{n+1}$ into a product of $n+1$ transpositions,
6. the number of critical states in the dollar game on K_{n+1}.

Generalized parking functions

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a sequence of positive integers. An $\mathbf{x}$-*parking function* is a sequence $(a_1, \ldots, a_n)$ of positive integers whose non-decreasing rearrangement $b_1 \le \cdots \le b_n$ satisfies $b_i \le x_1 + \cdots + x_i$.

Ordinary parking functions are the case $\mathbf{x} = (1, \ldots, 1)$.

Motivation

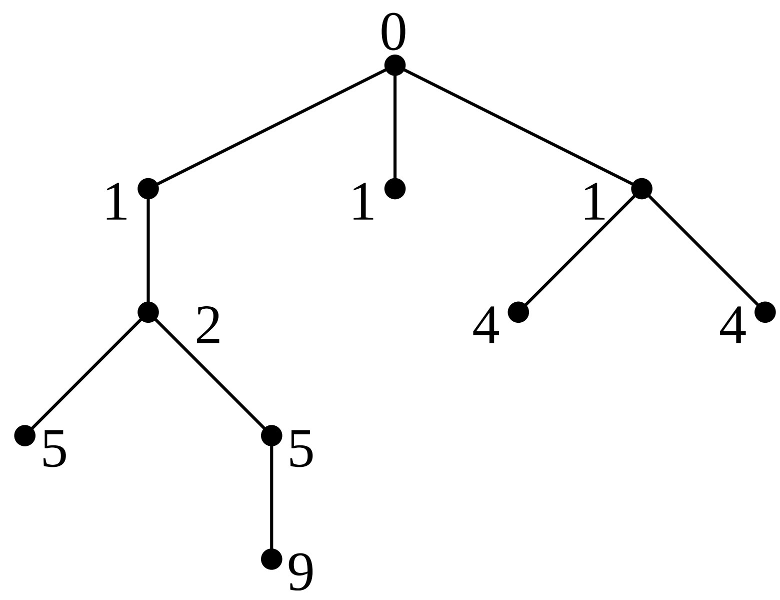
A parking function $(a_1, \ldots, a_n)$ is said to be *k-leading* if $a_1 = k$. Let $p_{n,k}$ denote the number of k-leading parking functions of length n. Foata and Riordan derived the generating function for $p_{n,k}$ algebraically,

$$\sum_{k=1}^{n} p_{n,k} x^k = \frac{x}{1-x} \left(2(n+1)^{n-2} - \sum_{k=1}^{n} \binom{n-1}{k-1} k^{k-2} (n-k+1)^{n-k-1} x^k \right).$$

Main results: We provide a unified combinatorial approach to the enumeration of $(a, b, \ldots, b)$-parking functions by their leading terms, which include the cases $\mathbf{x} = (1, \ldots, 1)$, $(a, 1, \ldots, 1)$, and $(b, \ldots, b)$.

Tree structures for parking functions

For example, consider α = (1,4,2,9,1,5,1,5,4). The non-decreasing re-arrangement of α is (1,1,1,2,4,5,5,9).



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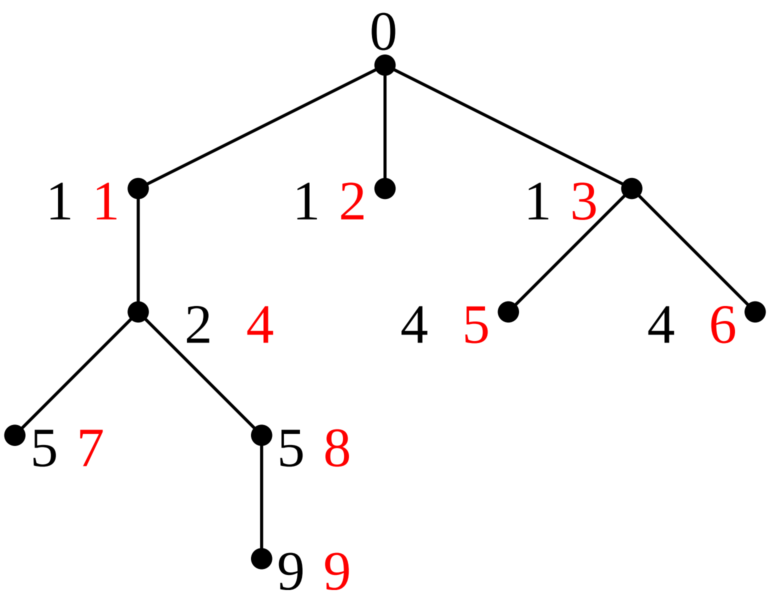
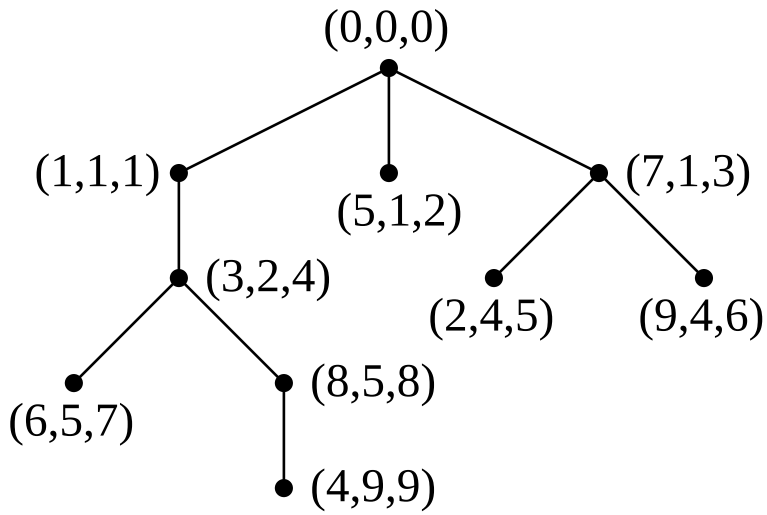


The breadth-first search (BFS) order of α is $\pi_\alpha =$ (1,5,4,9,2,7,3,8,6).

Triplet-labeled rooted trees

We associate $\alpha = $ (1,4,2,9,1,5,1,5,4) with a triplet-labeled rooted tree. The triplets are columns of the following array.

i	1	2	3	4	5	6	7	8	9
a_i	1	4	2	9	1	5	1	5	4
π_α	1	5	4	9	2	7	3	8	6



Definition of triplet-labeled rooted trees

Given a parking function $\alpha = (a_1, \ldots, a_n)$, for $1 \le i \le n$, we define

$$\pi_\alpha(i) = \text{ Card}\{a_j \in \alpha |\text{ either } a_j < a_i,\ \text{ or } a_j = a_i \text{ and } j < i\}. \qquad (1)$$

Note that $(\pi_\alpha(1), \ldots, \pi_\alpha(n))$ is a permutation of $[n] := \{1, \ldots, n\}$.

We associate α with a *triplet-labeled rooted tree* T_α whose vertex set is

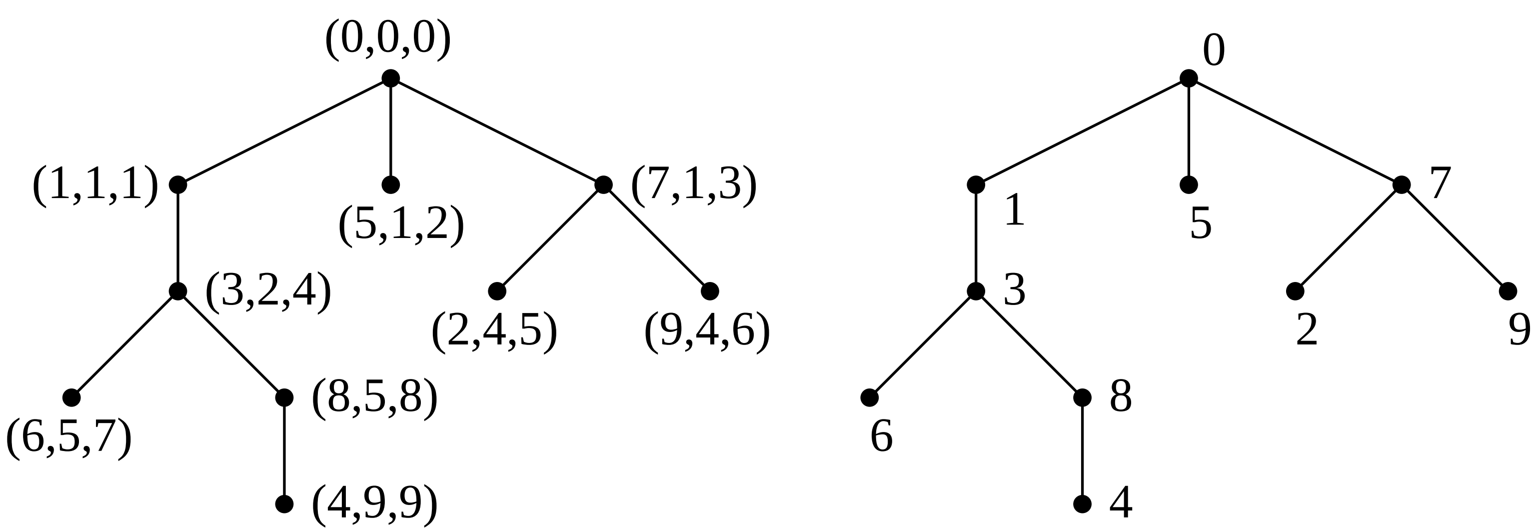
$$\{(0,0,0)\} \cup \{(i, a_i, \pi_\alpha(i)) | \, a_i \in \alpha\}.$$

Let T_α be rooted at $(0,0,0)$. For any two vertices $u = (i, a_i, \pi_\alpha(i))$ and $v = (j, a_j, \pi_\alpha(j))$, u is a child of v if $a_i = \pi_\alpha(j) + 1$.

The bijection ψ

Let $\mathcal{P}_n$ denote the set of parking functions of length n. Let $\mathcal{T}_n$ denote the set of labeled trees on the vertex set $[0, n]$.

The mapping $\psi : \mathcal{P}_n \to \mathcal{T}_n$ is defined as follows. $\psi(\alpha)$ is the same as the triplet-labeled rooted tree T_α associated with α with vertices labeled by the first entries of the triplets.



The inverse ψ^{-1}

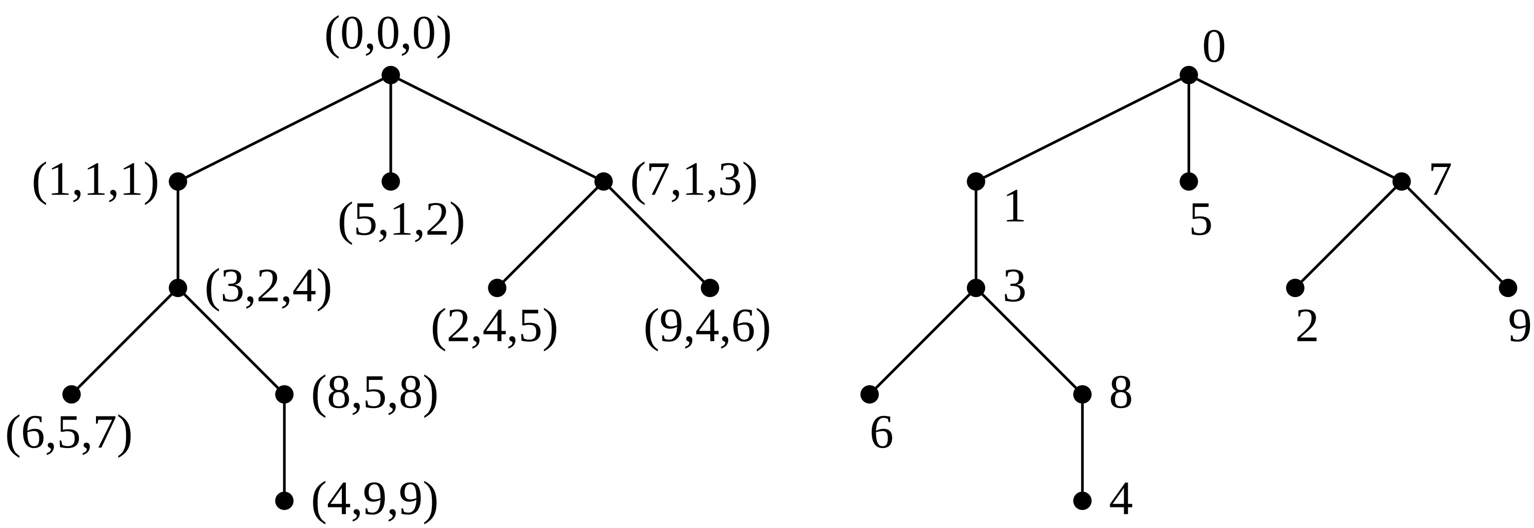
To describe ψ^{-1}, for each $T \in \mathcal{T}_n$, we express T in a form, called *canonical form*. Let T be rooted at 0. If a vertex of T has more than one child then the labels of these children are increasing from left to right.

To find ψ^{-1}, we associate the vertex $0 \in T$ with the triplet $(0, 0, 0)$, and for $1 \le i \le n$, associate the vertex $i \in T$ with a triplet (i, p_i, q_i), where p_i and q_i are determined by the following algorithm.

Algorithm A.

1. Traverse T from the root by a breadth-first search and label the third entries q_i of the vertices from 0 to n.

2. For any two vertices $u = (i, p_i, q_i)$ and $v = (j, p_j, q_j)$, if u is a child of v then $p_i = q_j + 1$.



A bijective result

Theorem. *For $1 \leq k \leq n-1$, there is a bijection between the set of k-leading parking functions α of length n that satisfy at least one of the two conditions*

(i) α *has more than one term equal to* k,

(ii) α *has at least* k *terms less than* k,

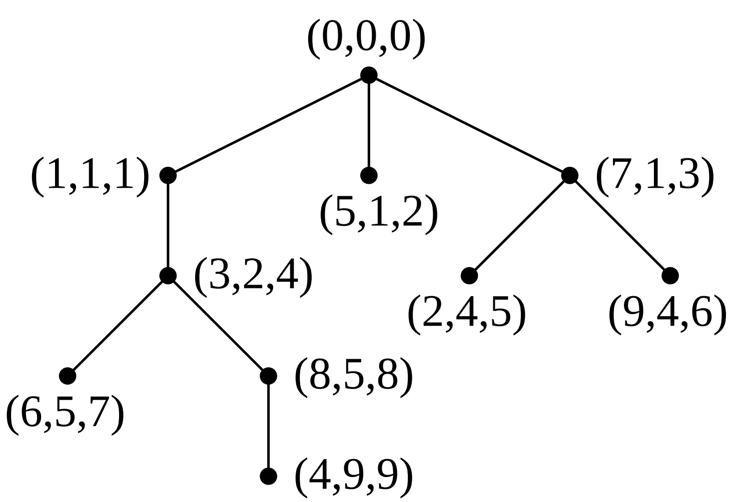
and the set of $(k+1)$-leading parking functions of length n.

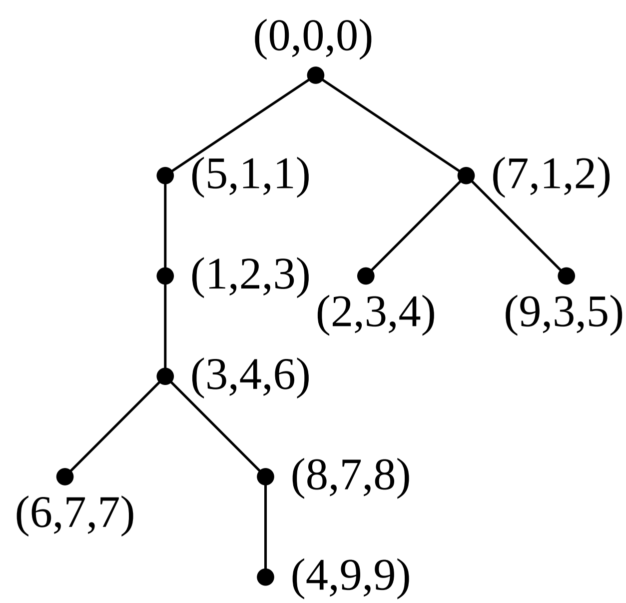
Example

On the left is the tree T_α associated with $\alpha =$ (1,4,2,9,1,5,1,5,4).

Let $u = (1,1,1)$. Note that $v = (5,1,2)$ is the first vertex of $T_\alpha - T(u)$ that is visited by a BFS.

On the right is the tree $\phi(T_\alpha)$. The corresponding 2-leading parking function is (2,3,4,9,1,7,1,7,3).



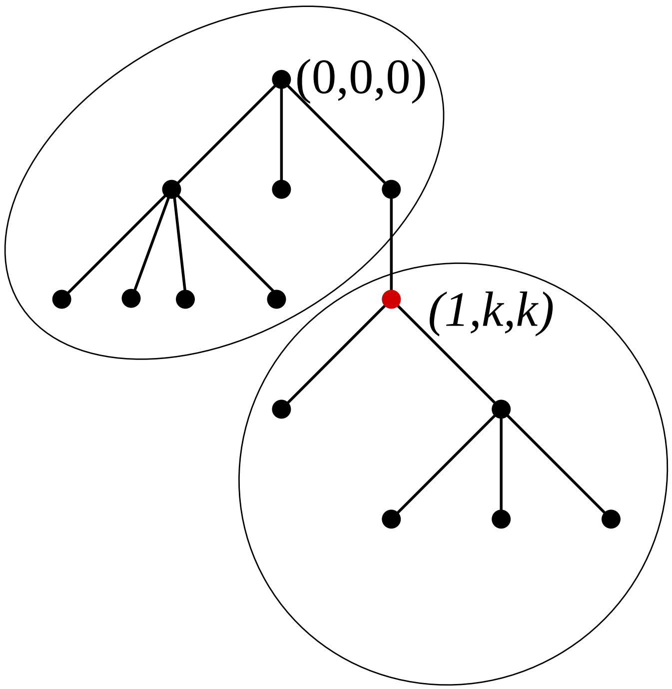


Enumeration of parking functions by leading terms

Let $p_{n,k}$ denote the number of k-leading parking functions of length n. By the previous bijective result, we derive the following recurrence relations.

Theorem. *For $1 \leq k \leq n-1$, we have*

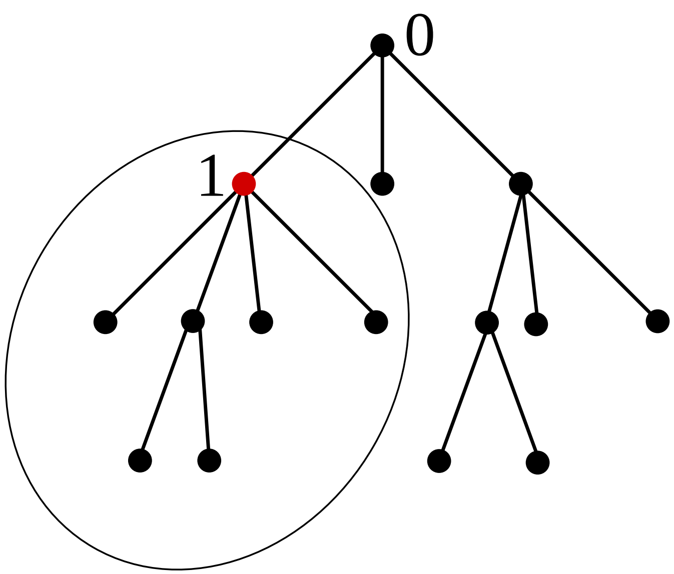
$$p_{n,k} - p_{n,k+1} = \binom{n-1}{k-1} k^{k-2} (n-k+1)^{n-k-1}.$$



The initial condition

In order to evaluate $p_{n,k}$ by the above recurrence relations, we derive the initial condition.

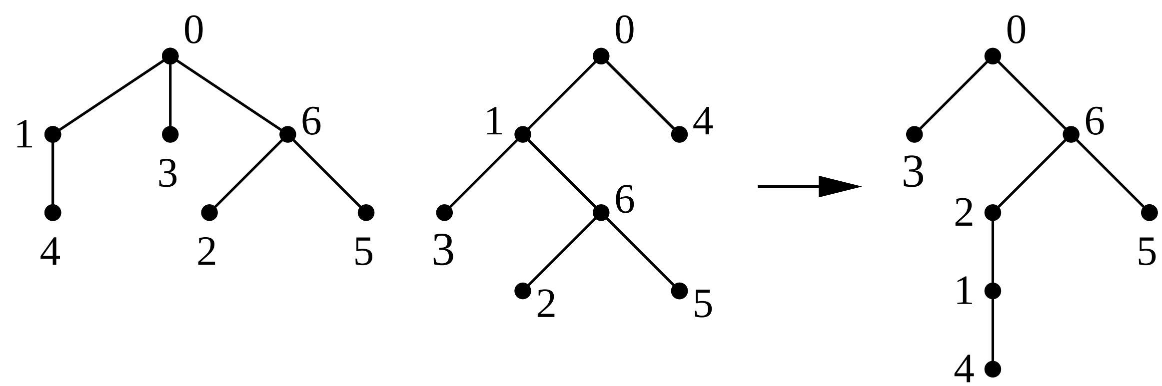
Theorem. $p_{n,1} = 2(n+1)^{n-2}$.



An interesting two-to-one correspondence

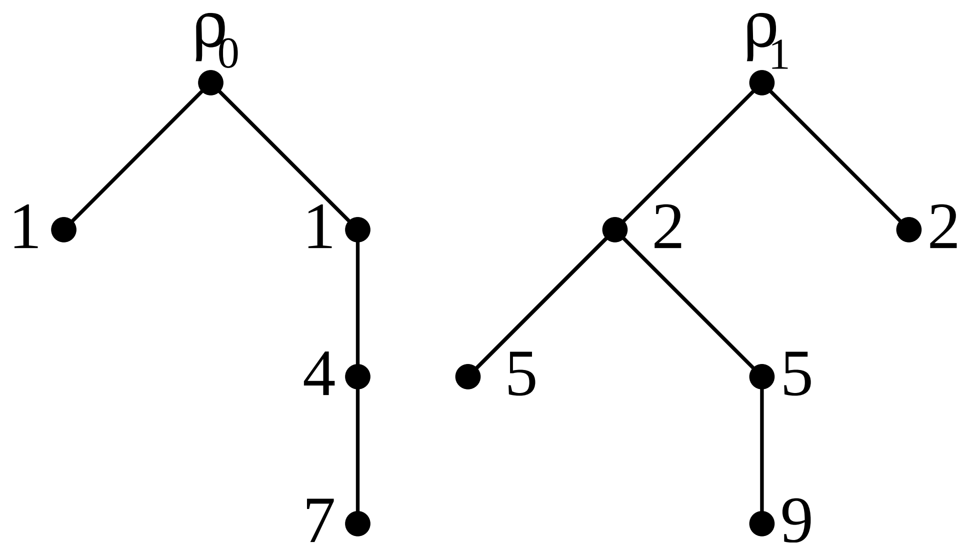
Theorem. *If n is even, then there is a two-to-one correspondence between the set of 1-leading parking functions of length n and the set of $(\frac{n}{2}+1)$-leading parking functions of length n.*

For example, take $n = 6$. On the left are the trees T corresponding to the parking functions $(1,4,1,2,4,1)$ and $(1,5,2,1,5,2)$. On the right is the tree corresponding to $(4,3,1,6,3,1)$.



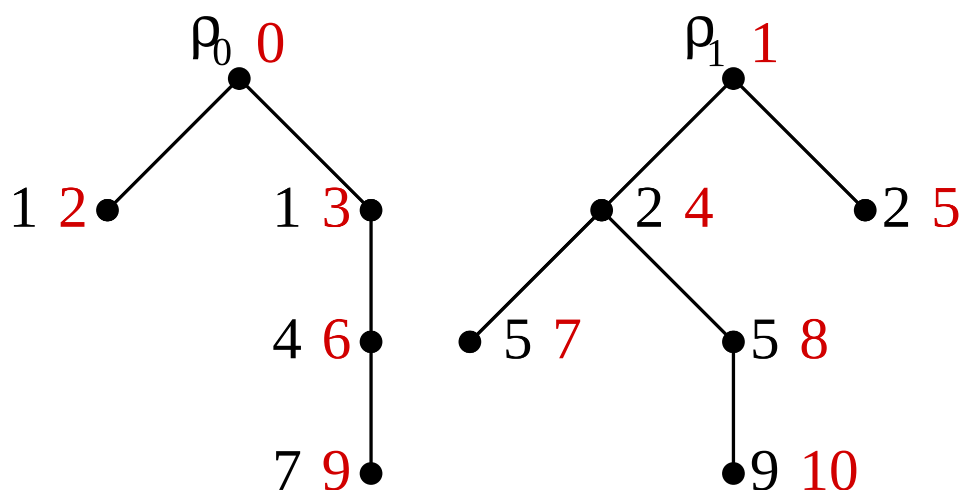
Forest structures for the $(a, 1, \ldots, 1)$-parking functions

Consider the $(2, \overline{1})$-parking function α = (2,5,9,1,5,7,2,4,1). The non-decreasing rearrangement is (1,1,2,2,4,5,5,7,9).



Forest structures for the $(a, 1, \ldots, 1)$-parking functions

Consider the $(2, \overline{1})$-parking function α = (2,5,9,1,5,7,2,4,1). The non-decreasing rearrangement is (1,1,2,2,4,5,5,7,9).



The breadth-first search (BFS) order of α is $\tau_\alpha =$ (4,7,10,2,8,9,5,6,3).

The BFS order τ_α

Given an $(a, \overline{1})$-parking function $\alpha = (a_1, \ldots, a_n)$, we define the permutation $(\pi_\alpha(1), \ldots, \pi_\alpha(n))$ as in (1), i.e.,

$$\pi_\alpha(i) = \text{ Card}\{a_j \in \alpha | \text{ either } a_j < a_i, \text{ or } a_j = a_i \text{ and } j < i\},$$

and define

$$\tau_\alpha(i) = \pi_\alpha(i) + a - 1, \text{ for } 1 \leq i \leq n.$$

Triplet-labeled rooted forests

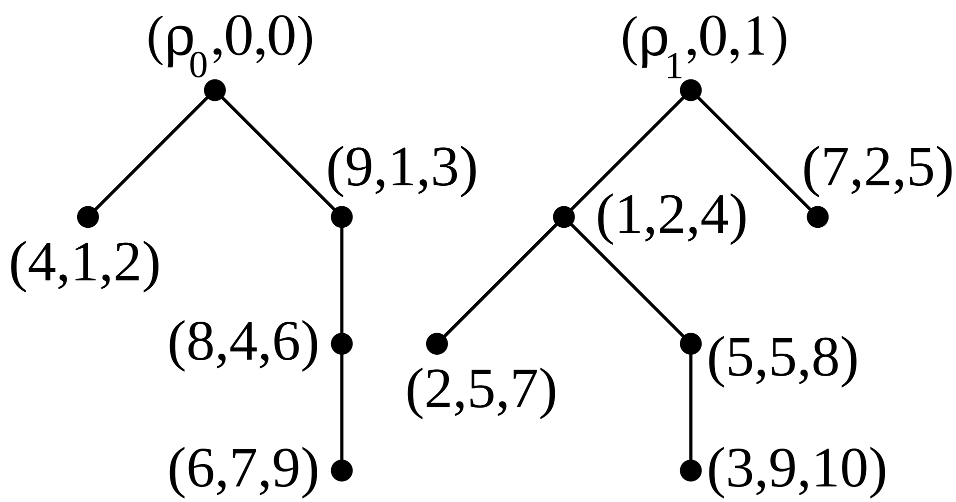
We associate α with an a-component forest F_α, called *triplet-labelled rooted forests*. The vertex set is

$$\{(i, a_i, \tau_\alpha(i)) |\, a_i \in \alpha\} \cup \{(\rho_i, 0, i) |\, 0 \leq i \leq a-1\}.$$

Let $(\rho_0, 0, 0), \ldots, (\rho_{a-1}, 0, a-1)$ be the roots of distinct trees. For any two vertices $u_1 = (x_1, y_1, z_1)$ and $u_2 = (x_2, y_2, z_2)$, u_2 is a child of u_1 if $y_2 = z_1 + 1$.

Example

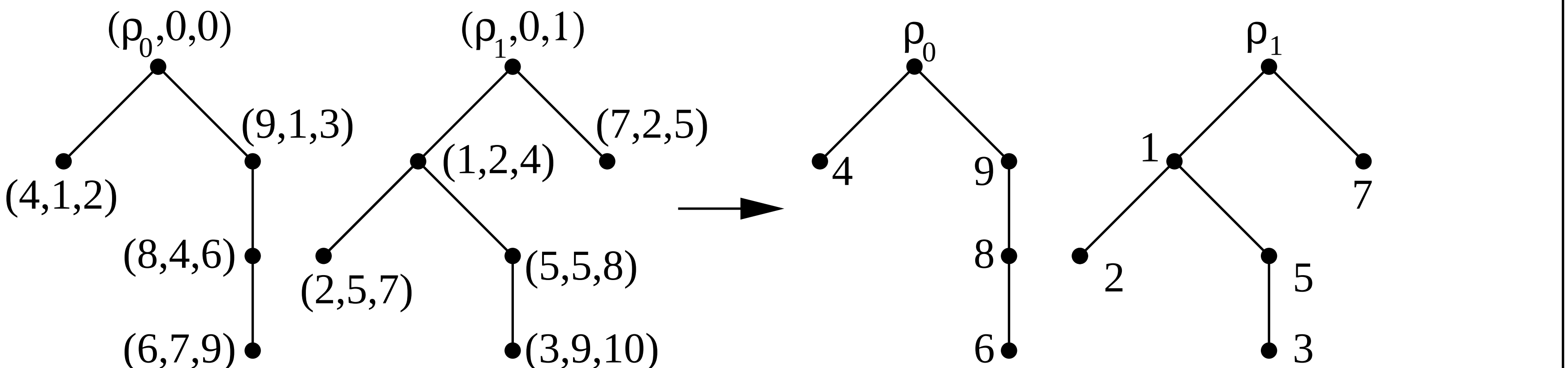
Consider the $(2,\overline{1})$-parking function $\alpha=$ (2,5,9,1,5,7,2,4,1). We have the permutation $(\pi_\alpha(1),\ldots,\pi_\alpha(n))=$ (3,6,9,1,7,8,4,5,2) and the sequence $(\tau_\alpha(1),\ldots,\tau_\alpha(n))=$ (4,7,10,2,8,9,5,6,3). The triplet-labelled rooted forest associated with α is shown below.



The bijection φ

Let $\mathcal{P}_n(a,\overline{1})$ denote the set of $(a,\overline{1})$-parking functions of length n and let $\mathcal{F}_n(a,\overline{1})$ denote the set of a-component rooted forests on the set $\{\rho_0,\ldots,\rho_{a-1}\} \cup [n]$ with roots $\rho_0,\ldots,\rho_{a-1}$. The bijection $\varphi : \mathcal{P}_n(a,\overline{1}) \to \mathcal{F}_n(a,\overline{1})$ is a generalization of ψ.

For example,



Another bijective result

Theorem. *For $a \leq k \leq a+n-2$, there is a bijection between the set of k-leading $(a, \overline{1})$-parking functions α of length n that satisfy at least one of the two conditions*

(i) α has more than one term equal to k,

(ii) α has at least $k-a+1$ terms less than k,

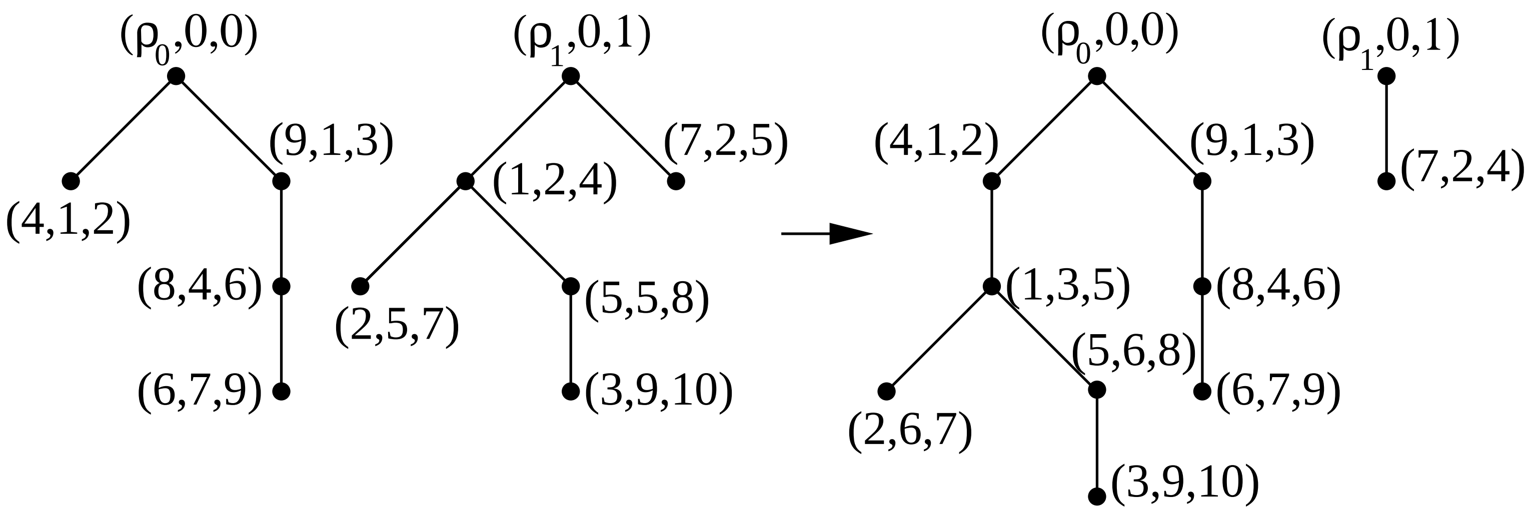
and the set of $(k+1)$-leading $(a, \overline{1})$-parking functions of length n.

Example

On the left is forest F associated with (2,5,9,1,5,7,2,4,1).

Let $u = (1, 2, 4)$. Note that $v = (4, 1, 2)$ is the second vertex of $F - T(u)$ that is visited by a BFS.

On the right is the forest $\phi(F)$. The corresponding 3-leading parking function is (3,6,9,1,6,7,2,4,1).



Enumeration

Theorem. *For $a \le k \le a+n-2$, we have*

$$p_{n,k}^{(a,\overline{1})} - p_{n,k+1}^{(a,\overline{1})} = \binom{n-1}{k-a} a k^{k-a-1} (n-k+a)^{n-k+a-2}. \qquad (2)$$

Theorem. *If $1 \le k \le a$, then the number of k-leading $(a,\overline{1})$-parking function of length n is $(a+1)(a+n)^{n-2}$, which is independent of k.*

The case of $(a, b, \ldots, b)$-parking functions

A *rooted b-forest* is a labeled rooted forest with edges colored with the colors $0, \ldots, b-1$.

Consider a sequence $(S_0, \ldots, S_t)$ of rooted b-forests on $[n]$ such that

(i) each S_i is a rooted b-forest,

(ii) S_i and S_j are disjoint if $i \neq j$, and

(iii) the union of the vertex sets S_i ($1 \leq i \leq t$) is $[n]$.

Let $\widehat{S}_i$ denote the rooted tree obtained by connecting the roots of S_i to a new root vertex ρ_i, where the edges incident to ρ_i are not colored with any color, denoted by -1 for such an edge.

Extended b-forests

$\mathcal{F}_n(a,\bar{b})$ denoted the set of a-component rooted forests of the form $(\widehat{S_0},\ldots,\widehat{S_{a-1}})$, where $(S_0,\ldots,S_{a-1})$ is a sequence of rooted b-forest on $[n]$. We call members of $\mathcal{F}_n(a,\bar{b})$ *extended b-forests*.

Let $\kappa(i)$ denote the color of the edge that connects the vertex i and its parent.

Reduction of $(a, \overline{b})$-parking functions

Let $\mathcal{P}_n(a, \overline{b})$ denote the set of $(a, \overline{b})$-parking functions of length n.

Given an $\alpha = (a_1, \ldots, a_n) \in \mathcal{P}_n(a, \overline{b})$, we associate α with two sequences $\beta = (p_1, \ldots, p_n)$ and $\gamma = (r_1, \ldots, r_n)$, where

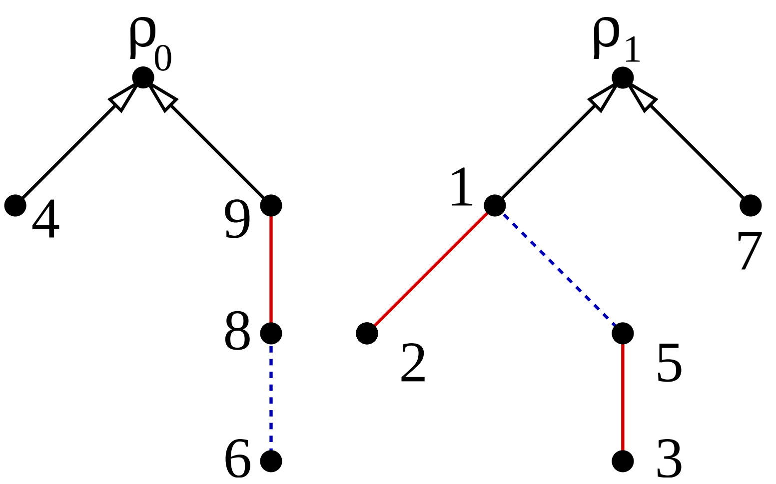
$$p_i = \begin{cases} a_i & \text{if } a_i \le a, \\ \left\lceil \dfrac{a_i - a}{b} \right\rceil + a & \text{otherwise;} \end{cases}$$

$$r_i = \begin{cases} -1 & \text{if } a_i \le a, \\ b(p_i - a) - a_i + a & \text{otherwise.} \end{cases}$$

One can verify that β is an $(a, \overline{1})$-parking function of length n and $\gamma \in [-1, b-1]^n$ with $r_i = -1$ whenever $a_i \le a$.

The bijection $\varphi_b : \mathcal{P}_n(a,\bar{b}) \to \mathcal{F}_n(a,\bar{b})$

For example, take $a = 2$ and $b = 2$. For the $\alpha = (2,7,15,1,8,12,2,5,1)$, we have the associated pair (β,γ), where $\beta = (2,5,9,1,5,7,2,4,1)$ and $\gamma = (-1,1,1,-1,0,0,-1,1,-1)$. To establish the mapping φ_b, we first locate the labeled rooted forest $\varphi(\beta) \in \mathcal{F}_n(a,\bar{1})$ and then define φ_b by carrying α into $\varphi(\beta)$ with the edge-coloring $\kappa(i) = r_i$, for $1 \leq i \leq n$.



The enumeration

Theorem. *Let $p_{n,m}^{(a,\overline{b})}$ denote the number of m-leading $(a,\overline{b})$-parking functions of length n.*

1. *For $0 \le k \le n-2$ and $bk+a+1 \le i,j \le b(k+1)+a$,*

$$p_{n,i}^{(a,\overline{b})} = p_{n,j}^{(a,\overline{b})}.$$

2. *For $0 \le k \le n-2$,*

$$p_{n,bk+a}^{(a,\overline{b})} - p_{n,bk+a+1}^{(a,\overline{b})} = \binom{n-1}{k} ab^{n-k-1}(a+bk)^{k-1}(n-k)^{n-k-2}.$$

3. *For $1 \le m \le a$,*

$$p_{n,m}^{(a,\overline{b})} = \sum_{j=0}^{n-1} \binom{n-1}{j} ab^{n-j-1}(a+bj)^{j-1}(n-j)^{n-j-2}.$$