Pooling Designs and Pooling Spaces

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d-disjunct matrix

Definition 0.1. An $n \times t$ matrix M over $\{0, 1\}$ is d-disjunct if $d < t$ and for any one column j and any other d columns $j_1, j_2, \ldots, j_d$, there exists a row i such that $M_{ij} = 1$ and $M_{ij_s} = 0$ for $s = 1, 2, \ldots, d$.

Example 0.2. A 2-disjunct matrix $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Relation to Pooling Design

A 4×6 1-disjunct matrix to detect the infected item $\mathbf{C}$ from $\{A, B, \mathbf{C}, D, E, F\}$:

$$\begin{pmatrix} \text{Tests/Items} & A & B & \mathbf{C} & D & E & F & & \text{Output} \\ \text{One} & 1 & 1 & 1 & 0 & 0 & 0 & \rightarrow & 1 \\ \text{Two} & 1 & 0 & 0 & 1 & 1 & 0 & \rightarrow & 0 \\ \text{Three} & 0 & 1 & 0 & 1 & 0 & 1 & \rightarrow & 0 \\ \text{Four} & 0 & 0 & 1 & 0 & 1 & 1 & \rightarrow & 1 \end{pmatrix}$$

Relation to Pooling Design (conti.)

If the size of defected items at most d, then a d-disjunct matrix works for finding the defected items.

Why?

Reason 1. All the subsets of the set of items with size at most d have different outputs.

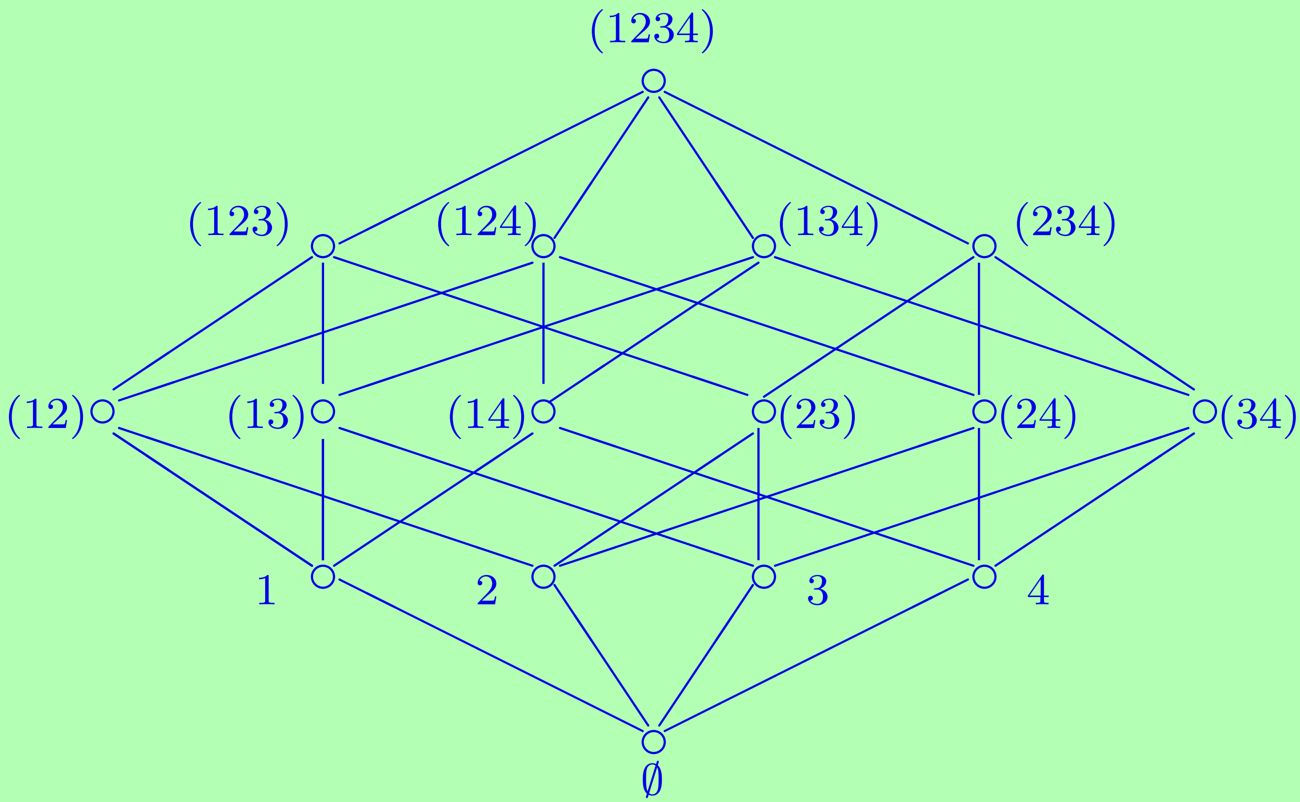
Reason 2. The tests with 0 outputs determine all the non-infected items.

Reason 3. The infected columns of are exactly those columns contained in the output vector (view vectors as subsets of $[n]$).

Construct d-disjunct matrices

Theorem 0.3. *(Macula 1996) Let* $[m] := \{1, 2, \ldots, m\}$*. The incident matrix* W_{dk} *of* d*-subsets and* k*-subsets of* $[m]$ *is an* $\begin{pmatrix} m \\ d \end{pmatrix} \times \begin{pmatrix} m \\ k \end{pmatrix}$ d*-disjunct matrix.*

The subsets of $[m]$ when $m = 4$



$W_{d,k}$ when $m = 4$

$$
\begin{array}{c@{\quad}c}
\frac{2\text{−subsets}}{1\text{−subsets}} & \begin{array}{cccccc} (12) & (13) & (14) & (23) & (24) & (34) \end{array} \\
\begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array} & \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}
\end{array}
$$

(d, s)-disjunct matrix

Definition 0.4. An $n \times t$ matrix M over $\{0, 1\}$ is (d, s)-disjunct if $d < t$ and for any one column j and any other d columns $j_1, j_2, \ldots, j_d$, there exist s rows $i_1, i_2, \ldots, i_s$ such that $M_{i_u j} = 1$ and $M_{i_u j_v} = 0$ for $u = 1, 2, \ldots, s$ and $v = 1, 2, \ldots, d$.

A (d, s)-disjunct matrix can be used to construct a pooling design that can find the set of defected item of size at most d with $\lfloor \frac{s-1}{2} \rfloor$ errors allowed in the output.

As an error-correcting code

Remark 0.5. *Let* M *be an* $n \times t$ (d, s)*-disjunct matrix over* $\{0, 1\}$*. Let* C *denote the set consisting of all the boolean sum of at most* d *columns of* M*. Then* $C \subseteq F_2^n$ *has cardinality* $\begin{pmatrix} t \\ 0 \end{pmatrix} + \begin{pmatrix} t \\ 1 \end{pmatrix} + \cdots + \begin{pmatrix} t \\ d \end{pmatrix}$ *and minimum distance* s*.*

Decoding algorithm

Theorem 0.6. *(Huang and Weng 2003) Let* M *be an* $n \times t$ (d, s)*-disjunct matrix over* $\{0, 1\}$*. Suppose the output vector* O *has at most* $\lfloor \frac{s-1}{2} \rfloor$ *errors. Then a column of* M *with at most* $\lfloor \frac{s-1}{2} \rfloor$ *elements not in* O *is an infected column.*

Example of (d, s)-disjunct matrix

Theorem 0.7. *(Huang and Weng 2004) Macula's d-disjunct matrix W_{dk} is $(d-1, k-d+1)$-disjunct.*

Posets

Definition 0.8. A poset P is ranked if there exists a function $\text{rank} : P \to \mathbb{N} \cup \{0\}$ such that for all elements $x, y \in P$,

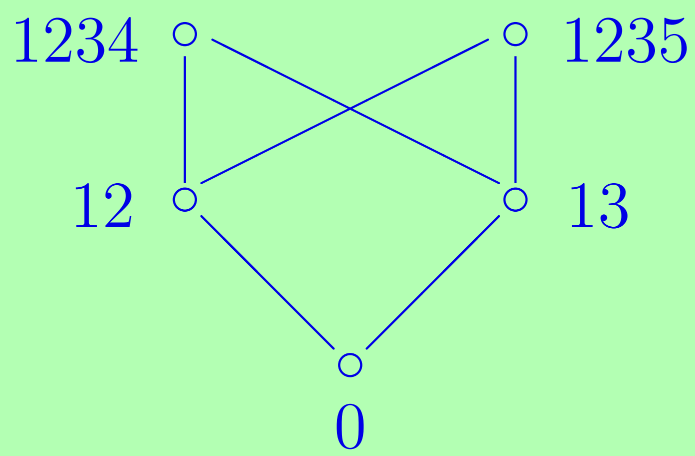
$$y \text{ covers } x \Rightarrow \text{rank}(x) - \text{rank}(y) = 1.$$

Let P_i denote the elements of rank i in P. P is atomic if each elements w is the least upper bound of the set $P_1 \cap \{y \leq w | y \in P\}$.

Pooling Spaces

Definition 0.9. A pooling space is a ranked poset P that the for each element $w \in P$ the subposet induced on $w^+ := \{y \geq w | y \in P\}$ is atomic.

A Nonexample of Pooling Spaces



Every interval in P is atomic, but P is not a pooling space.

More on Pooling Spaces

Theorem 0.10. *Let P be a ranked semi-lattice. Suppose each interval in P is atomic. Then P is a pooling space.*

d-disjunct matrices in Pooling Spaces

Theorem 0.11. *(Huang and Weng 2004) Let P be a pooling space. Then the incident matrix P_{dk} of rank d elements P_d and rank k elements P_k is a d-disjunct matrix. In fact, P_{dk} is $(\mathrm{d}', s_{\mathrm{d}'})$-disjunct matrix for some large integer $s_{\mathrm{d}'}$ depending on $\mathrm{d}' \leq \mathrm{d}$ and P.*

Examples of Pooling Spaces

Hamming matroids, the attenuated spaces, quadratic polar spaces, alternating polar spaces, quadratic polar spaces (two types), Hermitian polar spaces (two types). These are called quantum matroids.

Combinatorial Geometry

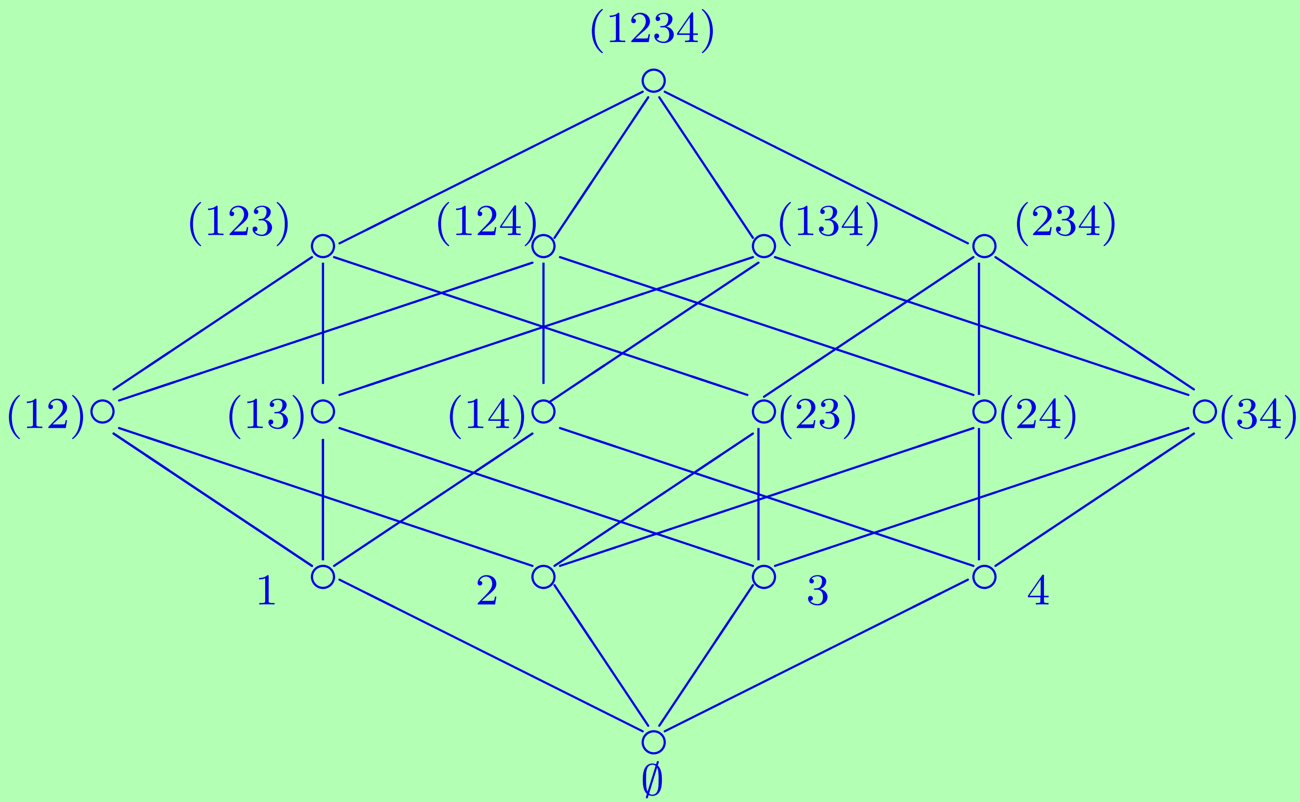
Definition 0.12. A combinatorial geometry is a pair $(X, \mathcal{F})$ where X is a set of points and where $\mathcal{F}$ is a family of subsets of X called flats such that

(1) $\mathcal{F}$ is closed under intersection;

(2) $\emptyset$, X, $\{x\} \in \mathcal{F}$ for all $x \in X$;

(3) For $E \in \mathcal{F}$, $E \neq X$, the flats that cover E in $\mathcal{F}$ partition the remaining points.

An example of combinatorial geometry



Combinatorial Geometry is a Pooling Space

Theorem 0.13. *Let P be a combinatorial geometry. Then $(P, \subseteq)$ is a pooling space.*

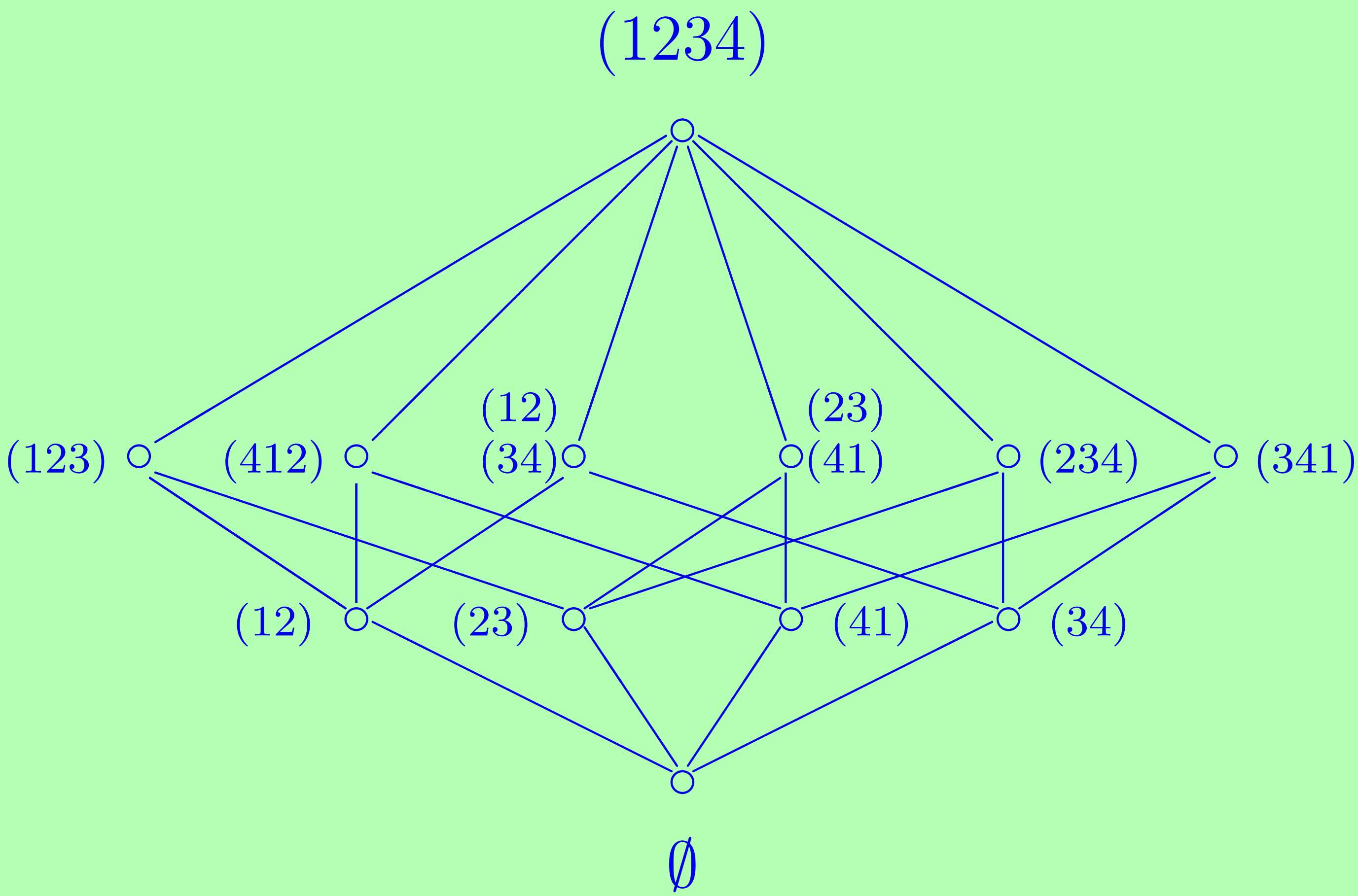
Other Examples of Combinatorial Geometries

1. All the partitions of a finite set X ordered by refinement;

2. Fix a graph G. The partitions of the vertices of G with connected blocks, ordered by refinement.

Note. 1 is the special case of 2 with G the complete graph.

Connected partitions of the 4-cycle



Atomic Graphs

Definition 0.14. Let G be a graph. For two vertices x, y in G, let $C(x, y)$ denote the set of neighbors of x in a geodesic from x to y. G is atomic with respect to x if

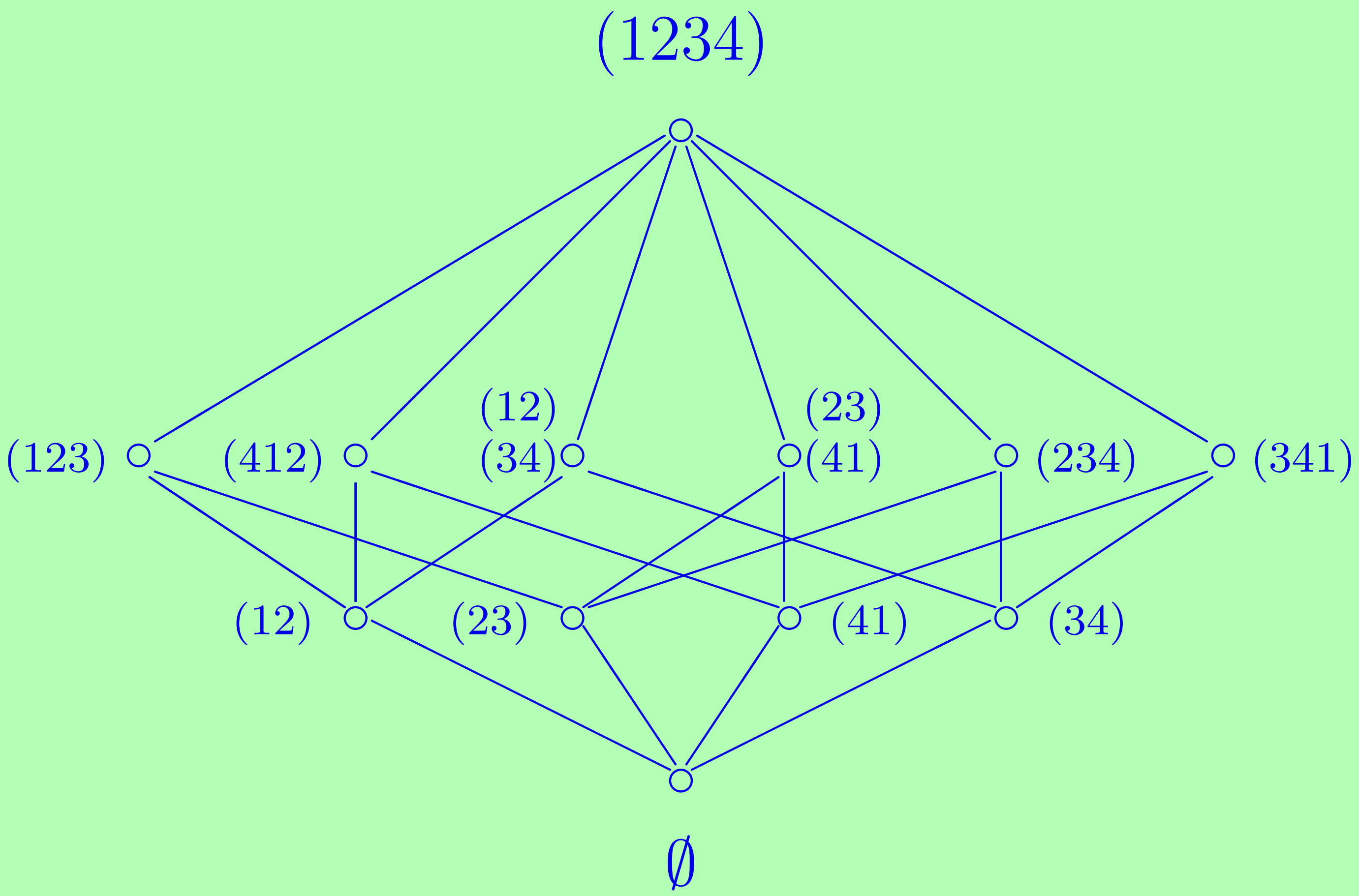
$$C(x, y) = C(x, z) \Rightarrow y = z \qquad (x, y, z \in G).$$

G is atomic if G is atomic with respect to all vertices of G.

Examples of Atomic Graphs

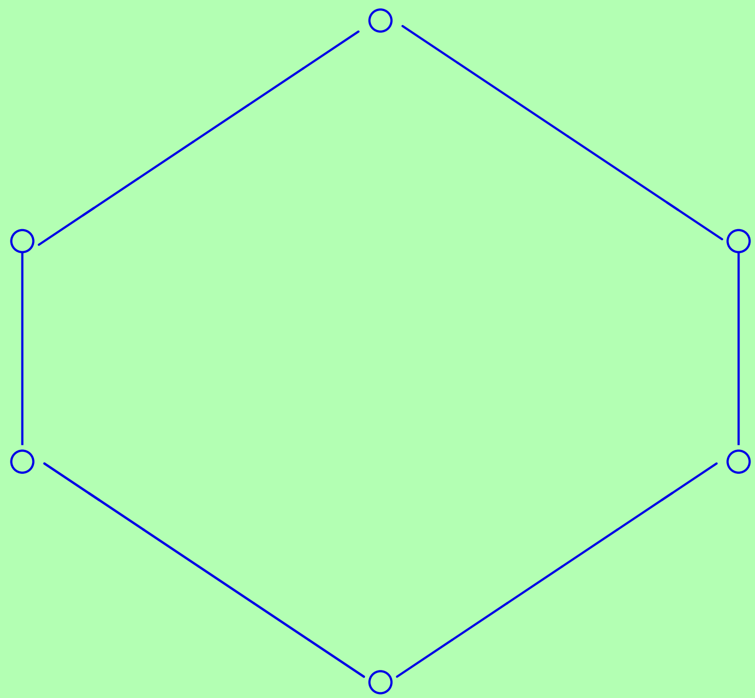
Let P be a pooling space. Then the graph with vertex set w^+ and edge defined by covering relation in P is atomic with respect to w for any $w \in P$.

The graph is atomic w.r.t. to $\emptyset$ but not (123)

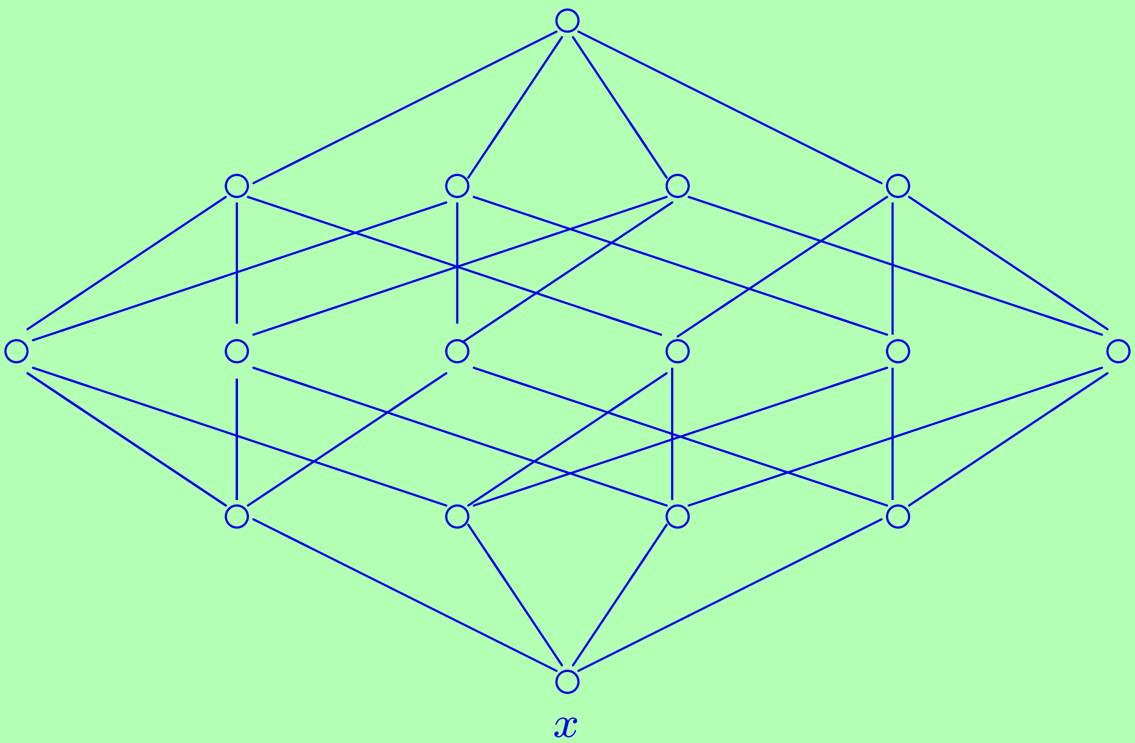


Conjecture

The class of cycles is the only class of Distance-regular graphs with unbounded diameters that is not atomic.



The distance-regular graph 4-cube



In general, n-cube is atomic.

Hard open question

Classify all atomic graphs

The end

Thank You!