A Factorization for Multivariate Formal Power Series and its Applications

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$$E(z) = 1 + e_1 z + e_2 z^2 + \cdots \in 1 + z\mathbb{C}[[z]],$$

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by the equations

$$H(z) = \frac{1}{R(z)}, \tag{1}$$
$$E(z) = R(-z), \tag{2}$$

and

$$P(z) = -z\frac{R'(z)}{R(z)} = z\frac{H'(z)}{H(z)}. \tag{3}$$

Symmetric Functions

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Assume that $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{C}$ are the roots of $F(z)$. We can write

$$F(z) = \prod_{i=1}^{r} (z - \alpha_i),$$

or equivalently

$$R(z) = \prod_{i=1}^{r} (1 - \alpha_i z). \tag{4}$$

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The *nth complete homogeneous* symmetric function in the roots of $F(z)$ is

$$h_n := \sum_{1 \le i_1 \le \cdots \le i_n \le r} \alpha_{i_1} \cdots \alpha_{i_n}.$$

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$$e_n := \sum_{1 \le i_1 < \cdots < i_n \le r} \alpha_{i_1} \cdots \alpha_{i_n}.$$

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The corresponding generating functions are

$$H(z) := \sum_{n\geq 0} h_n z^n = \prod_{i=1}^{k} \frac{1}{(1-\alpha_i z)},$$

$$E(z) := \sum_{n\geq 0} e_n z^n = \prod_{i=1}^{k} (1+\alpha_i z),$$

$$P(z) := \sum_{n\geq 1} p_n z^n = \sum_{i=1}^{k} \frac{\alpha_i z}{(1-\alpha_i z)}.$$

Infinite Product

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Definition. The *order* of nonzero $F(z) \in \mathbb{C}[[z]]$ is

$$\mathrm{ord} F(z) = \text{the smallest } n \text{ such that } z^n \text{ has nonzero coefficient in } F(z).$$

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Proposition. *Let $F_n(z) \in \mathbb{C}[[z]]$ with $F_n(0) = 0$ for $n \geq 1$. Then*

$$\prod_{n \geq 1} (1 + F_n(z))$$

converges if and only if

$$\lim_{n \to \infty} \mathrm{ord} F_n(z) = \infty.$$

Factorization

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Theorem. *If* $R_n(z)$*, for all* $n \geq 1$ *are formal power series in* $\mathbb{C}[[z]]$ *with* $\mathrm{ord} R_n(z) = n$*, then there are unique* $C_n \in \mathbb{C}, n \geq 1$*, with*

$$R(z) = \prod_{n \geq 1} (1 + R_n(z))^{C_n}.$$

Möbius Inversion Theorem

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Definition. The *Möbius function* $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n = p_1 p_2 \cdots p_k, \\ 0 & \text{otherwise.} \end{cases}$$

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Möbius Inversion Theorem. *Let $\alpha(n)$ and $\beta(n)$ be arithmetic functions. Then*

$$\alpha(n) = \sum_{d|n} \beta(d) = \sum_{d|n} \beta(n/d), \qquad \textit{for all } n \geq 1.$$

if and only if

$$\beta(n) = \sum_{d|n} \mu(d)\alpha(n/d), \qquad \textit{for all } n \geq 1.$$

Type I

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Theorem. *Let* $R(z) = 1 + a_1 z + a_2 z^2 + \cdots \in 1 + z\mathbb{C}[[z]]$. *There are unique* $M_n \in \mathbb{C}, n \geq 1$, *with*

$$R(z) = \prod_{n \geq 1} (1 - z^n)^{M_n}. \tag{5}$$

Moreover, we have

$$p_n = \sum_{d|n} d M_d \quad \forall n \geq 1 \tag{6}$$

and

$$M_n = \frac{1}{n} \sum_{d|n} \mu(d) p_{n/d} \quad \forall n \geq 1. \tag{7}$$

Proof of Theorem (Type I)

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Proof: The first statement is clear because of Factorization Theorem. Now taking the logarithmic derivative on both sides of (5), and multiplying by $-z$ gives

$$-z\frac{R'(z)}{R(z)} = \sum_{n\geq 1} nM_n \frac{z^n}{1-z^n}.$$

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Comparing the coefficients on both sides, we get

$$p_n = \sum_{d|n} dM_d \quad \forall n \geq 1.$$

Finally, equation (7) follows by applying the Möbius Inversion Theorem to (6).

Let $R(z) = 1 - \alpha z$ where $\alpha \in \mathbb{P}$. We get $p_n = \alpha^n$ from equation (3). Hence, by equation (4) and (6), we have

$$1 - \alpha z = \prod_{n \geq 1} (1 - z^n)^{M_n}$$

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or equivalently

$$\frac{1}{1 - \alpha z} = \prod_{n \geq 1} \left(\frac{1}{1 - z^n} \right)^{M_n}, \qquad \text{where} \qquad M_n = \frac{1}{n} \sum_{d | n} \mu(d) \alpha^{n/d}.$$

Cyclotomic Identity

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Remark: It is worth noting that $M_n = \frac{1}{n} \sum_{d|n} \mu(d) \alpha^{n/d}$ is the number of primitive necklaces with n beads and α colors.

Congruence (Type I)

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Theorem. *The following three conditions are equivalent*

(i) $R(z) \in 1 + z\mathbb{Z}[[z]]$,

(ii) $M_n \in \mathbb{Z} \quad \forall n \geq 1$,

(iii) $\sum_{d|n} \mu(d) p_{n/d} \equiv 0 \pmod{n} \quad \forall n \geq 1$.

Let $R(z) = 1 - \alpha z$ where $\alpha \in \mathbb{P}$. We get $p_n = \alpha^n$ from equation (3). By Theorem, we have

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$$\begin{aligned} &\sum_{d|q} \mu(d)\alpha^{q/d} \\ =&\mu(1)\alpha^q + \mu(q)\alpha \end{aligned}$$

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$$\begin{aligned}
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=&\mu(1)\alpha^q + \mu(q)\alpha \\
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\end{aligned}$$

Fermat's Little Theorem

Let $R(z) = 1 - \alpha z$ where $\alpha \in \mathbb{P}$. We get $p_n = \alpha^n$ from equation (3). By Theorem, we have

$$\sum_{d|n} \mu(d)\alpha^{n/d} \equiv 0 \pmod{n}$$

Let $n = q$ be a prime, then we have

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Characterization (Type I)

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Theorem. *The following are equivalent:*

(i) $\exp\left(\sum_{n\geq 1} \frac{p_n}{n} z^n\right) \in 1 + z\mathbb{Z}[[z]]$,

(ii) $\sum_{d|n} \mu(d) p_{n/d} \equiv 0 \pmod{n}$ *for all* $n \geq 1$,

(iii) $\sum_{d|n} \alpha(d) p_{n/d} \equiv 0 \pmod{n}$ *for all* $n \geq 1$, *where* α *is an arithmetic function with* $\alpha(1) = \pm 1$ *and* $\sum_{d|n} \alpha(d) \equiv 0 \pmod{n}$ *for all* $n \geq 2$,

(iv) $p_{mq^s} \equiv p_{mq^{s-1}} \pmod{q^s}$ *for all primes* q *and* $m, s \in \mathbb{P}$.

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$$h_n = \mathcal{C}(n) = \frac{1}{n+1}\binom{2n}{n}$$

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and

$$\sum_{d|n} \mu(d) \binom{\frac{2n}{d}-1}{\frac{n}{d}-1} \equiv 0 \pmod{n}$$

where $\mathcal{C}(n)$ is the Catalan number.

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Let $H(z) = \frac{1-z-\sqrt{1-2z-3z^2}}{2z^2}$.

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Then

$$h_n = \mathcal{M}(n) = \frac{1}{n+1} \sum_i \binom{n+1}{i} \binom{n+1-i}{i+1}$$

$$p_n = \mathcal{CT}(n) = \sum_i \binom{n}{i} \binom{n-i}{i}$$

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and

$$\sum_{d|n} \mu(d) \mathcal{CT}(n/d) \equiv 0 \pmod{n}$$

where $\mathcal{M}(n)$ is the Motzkin number and $\mathcal{CT}(n)$ is the central trinomial coefficient.

Example 3

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$$h_n = \mathcal{S}(n) = \frac{1}{n}\sum_i 2^i \binom{n}{i}\binom{n}{i-1}$$

$$p_n = \frac{1}{2}[\mathcal{CD}(n) + \mathcal{CD}(n-1)]$$

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$$p_n = \frac{1}{2}[\mathcal{CD}(n) + \mathcal{CD}(n-1)]$$

and

$$\sum_{d|n} \mu(d)\frac{1}{2}[\mathcal{CD}(n/d) + \mathcal{CD}(n/d-1)] \equiv 0 \pmod{n}$$

where $\mathcal{S}(n)$ is the large Schröder number and $\mathcal{CD}(n)$ is the central Delannoy number.

Remark: The large Schröder number $\mathcal{S}(n)$ is the number of subdiagonal paths from $(0,0)$ to (n,n) consisting of steps east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ (sometimes called royal paths).

The central Delannoy number $\mathcal{CD}(n)$ is the number of paths from $(0,0)$ to (n,n) consisting of steps east $(1,0)$, north $(0,1)$, and northeast $(1,1)$.

Multivariate Setting

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Let $\mathbf{z} = \{z_1, ..., z_k\}$ be a set of commutative indeterminates and $F[[\mathbf{z}]] = F[[z_1, ..., z_k]]$.

Let boldface letters denote vectors

$\mathbf{z}^{\mathbf{n}} = z_1^{n_1} \cdots z_k^{n_k}$.

$\mathbb{N} = \{0, 1, 2, \cdots\}$

$S = \mathbb{N}^k \setminus \{\mathbf{0}\}$.

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$S = \mathbb{N}^k \setminus \{\mathbf{0}\}$.

Let $R(\mathbf{z}) = \sum_{\mathbf{n} \geq \mathbf{0}} a_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{C}[[\mathbf{z}]]$ with $R(\mathbf{0}) = 1$

Define

$$H(\mathbf{z}) = \sum_{\mathbf{n} \geq \mathbf{0}} h_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{C}[[\mathbf{z}]],$$

$$E(\mathbf{z}) = \sum_{\mathbf{n} \geq \mathbf{0}} e_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{C}[[\mathbf{z}]],$$

and

$$P(\mathbf{z}) = \sum_{\mathbf{n} \in S} p_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} \in \mathbb{C}[[\mathbf{z}]]$$

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by the equations

$$H(\mathbf{z}) = \frac{1}{R(\mathbf{z})}, \tag{8}$$

$$E(\mathbf{z}) = R(-\mathbf{z}), \tag{9}$$

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$$E(\mathbf{z}) = R(-\mathbf{z}), \tag{9}$$

and

$$P(\mathbf{z}) = -\frac{\sum_{i=1}^{k} z_i D_i R(\mathbf{z})}{R(\mathbf{z})} = \frac{\sum_{i=1}^{k} z_i D_i H(\mathbf{z})}{H(\mathbf{z})}. \tag{10}$$

Möbius Inversion Theorem (Multivariate Version)

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We use the notation $d|\mathbf{n}$ to mean that d divides all components n_i and write

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Theorem. *Let $\alpha(\mathbf{n})$ and $\beta(\mathbf{n})$ be functions defined on S. Then*

$$\alpha(\mathbf{n}) = \sum_{d|\mathbf{n}} \beta(\mathbf{n}/d), \qquad \text{for all } \mathbf{n} \in S.$$

if and only if

$$\beta(\mathbf{n}) = \sum_{d|\mathbf{n}} \mu(d)\alpha(\mathbf{n}/d), \qquad \text{for all } \mathbf{n} \in S.$$

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Theorem. *Let $R(\mathbf{z}) \in \mathbb{C}[[\mathbf{z}]]$ with $R(\mathbf{0}) = 1$. There are unique $M_{\mathbf{n}} \in \mathbb{C}, \mathbf{n} \in S$, with*

$$R(z) = \prod_{\mathbf{n} \in S} (1 - \mathbf{z}^{\mathbf{n}})^{M_{\mathbf{n}}}. \tag{11}$$

Moreover, we have

$$p_{\mathbf{n}} = \sum_{d|\mathbf{n}} \left| \frac{\mathbf{n}}{d} \right| M_{\mathbf{n}/d} \quad \forall \mathbf{n} \in S \tag{12}$$

and

$$M_{\mathbf{n}} = \frac{1}{|\mathbf{n}|} \sum_{d|n} \mu(d) p_{\mathbf{n}/d} \quad \forall \mathbf{n} \in S. \tag{13}$$

Congruence (MType I)

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Theorem. *The following three conditions are equivalent*

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(ii) $M_{\mathbf{n}} \in \mathbb{Z} \quad \forall \mathbf{n} \in S$

(iii) $\displaystyle\sum_{d|\mathbf{n}} \mu(d) p_{\mathbf{n}/d} \equiv 0 \pmod{|\mathbf{n}|} \quad \forall \mathbf{n} \in S.$

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and

$$\sum_{d|(m,n)} \mu(d) \left[\binom{m/d}{n/d} + \binom{m/d-1}{n/d-1} \right] \equiv 0 \pmod{m+n}$$

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$$p_{m,n} = \mathcal{D}(m, n) + \mathcal{D}(m-1, n-1)$$

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$$p_{m,n} = \mathcal{D}(m,n) + \mathcal{D}(m-1,n-1)$$

and

$$\sum_{d|(m,n)} \mu(d)\left[\mathcal{D}(m/d,n/d) + \mathcal{D}(m/d-1,n/d-1)\right] \equiv 0 \pmod{m+n}$$

where $\mathcal{D}(m,n)$ is the Delannoy number.

Example 3

Let $R(x, y) = 1 - x - y - xy$.

Then

$$h_{m,n} = \mathcal{D}(m, n) = \sum_i \binom{n}{i}\binom{m+n-i}{n} = \sum_i 2^i \binom{m}{i}\binom{n}{i}$$

$$p_{m,n} = \mathcal{D}(m, n) + \mathcal{D}(m-1, n-1)$$

and

$$\sum_{d|(m,n)} \mu(d)\left[\mathcal{D}(m/d, n/d) + \mathcal{D}(m/d-1, n/d-1)\right] \equiv 0 \pmod{m+n}$$

where $\mathcal{D}(m, n)$ is the Delannoy number.

Remark: The Delannoy number $\mathcal{D}(m, n)$ is the number of lattice paths from $(0, 0)$ to (m, n) consisting of steps east $(1, 0)$, north $(0, 1)$, and northeast $(1, 1)$.

Thank you :)