

## SPECTRAL RADIUS AND AVERAGE 2-DEGREE SEQUENCE OF A GRAPH

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In a simple connected graph, the average 2-degree of a vertex is the average degree of its neighbors. With the average 2-degree sequence and the maximum degree ratio of adjacent vertices, we present a sharp upper bound of the spectral radius of the adjacency matrix of a graph, which improves a result in [Y. H. Chen, R. Y. Pan and X. D. Zhang, Two sharp upper bounds for the signless Laplacian spectral radius of graphs, *Discrete Math. Algorithms Appl.* **3**(2) (2011) 185–191].

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### 1. Introduction

Let  $G = (V, E)$  be a simple connected graph with vertex set  $V(G)$ , edge set  $E(G)$  and  $n = |V(G)|$ . For any vertex  $v \in V(G)$ , let  $d_v$  denote the degree of  $v$ , define the *average 2-degree*  $M_v$  of  $v$  to be the average degree of the neighbors of  $v$ . In other words,  $M_v = \sum_{u \sim v} d_u / d_v$ , where  $u \sim v$  means vertices  $u$  and  $v$  are adjacent. Throughout the paper, label the vertices of  $G$  by  $1, 2, \dots, n$  such that  $M_1 \geq M_2 \geq \dots \geq M_n$ . Let  $A = (a_{ij})$  be the *adjacency matrix* of  $G$ , a binary matrix of order  $n$  such that for any pair  $i, j \in V(G)$ ,  $a_{ij} = 1$  if and only if  $i, j$  are adjacent in  $G$ . The *spectral radius*  $\rho(G)$  of  $G$  is the largest eigenvalue of its adjacency matrix; this parameter has been studied by many authors [1, 2, 5–9, 11–13] and can be

used to induce some other bounds such as the upper bounds of signless Laplacian eigenvalues [3, 4].

The following theorem is well-known and referred as Perron–Frobenius Theorem [10, Chap. 2].

**Theorem 1.1.** *If  $B$  is a nonnegative irreducible  $n \times n$  matrix with largest eigenvalue  $\rho(B)$  and row-sums  $r_1, r_2, \dots, r_n$ , then*

$$\rho(B) \leq \max_{1 \leq i \leq n} r_i$$

*with equality if and only if the row-sums of  $B$  are all equal.*

In this paper, we pay attention to the upper bounds of spectral radius of graphs in terms of the average 2-degree sequence. By setting  $B = U^{-1}AU$ , where  $U = \text{diag}(d_1, d_2, \dots, d_n)$ , the following fact is easily seen from Theorem 1.1.

**Theorem 1.2.**

$$\rho(G) \leq M_1$$

*with equality if and only if  $M_1 = M_2 = \dots = M_n$ .*

A graph for which equality holds in Theorem 1.2 is called *pseudo-regular* in [13].

In 2011 [3, Theorem 2.1], Chen, Pan and Zhang gave the following bound.

**Theorem 1.3.** *Let  $a := \max\{d_i/d_j \mid 1 \leq i, j \leq n\}$ . Then*

$$\rho(G) \leq \frac{M_2 - a + \sqrt{(M_2 + a)^2 + 4a(M_1 - M_2)}}{2},$$

*with equality if and only if  $G$  is pseudo-regular.*

We will show in Corollary 3.3 that Theorem 1.3 is indeed a generalization of Theorem 1.2. Moreover, we give the following theorem to generalize Theorem 1.3.

**Theorem 1.4.** *For any  $b \geq \max\{d_i/d_j \mid i \sim j\}$  and  $1 \leq \ell \leq n$ ,*

$$\rho(G) \leq \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b \sum_{i=1}^{\ell-1} (M_i - M_\ell)}}{2},$$

*with equality if and only if  $G$  is pseudo-regular.*

Note that Theorem 1.3 is a special case of Theorem 1.4 by taking  $b = a$  and  $\ell = 2$ . The proof of Theorem 1.4 is a subtle application of Perron–Frobenius Theorem. This idea was previously employed in [9, 11]. Indeed, our proof is an edited version of the proof of [9, Theorem 1.7].

We provide some examples of pseudo-regular graphs that are not regular in Example 2.1. The lowest upper bound among the choices of  $b$  and  $\ell$  is investigated in Sec. 3.

**2. Proof of Theorem 1.4**

**Proof of Theorem 1.4.** For each  $1 \leq i \leq \ell - 1$ , let  $x_i \geq 1$  be a variable to be determined later. Let  $U = \text{diag}(d_1x_1, \dots, d_{\ell-1}x_{\ell-1}, d_\ell, \dots, d_n)$  be a diagonal matrix of size  $n \times n$ . Consider the matrix  $B = U^{-1}AU$ . Note that  $A$  and  $B$  have the same eigenvalues. Let  $r_1, r_2, \dots, r_n$  be the row-sums of  $B$ . Then for  $1 \leq i \leq \ell - 1$  we have

$$\begin{aligned} r_i &= \sum_{k=1}^{\ell-1} \frac{1}{d_i x_i} a_{ik} d_k x_k + \sum_{k=\ell}^n \frac{1}{d_i x_i} a_{ik} d_k \\ &= \frac{1}{x_i} \sum_{k=1}^{\ell-1} (x_k - 1) a_{ik} \frac{d_k}{d_i} + \frac{1}{x_i} \sum_{k=1}^n a_{ik} \frac{d_k}{d_i} \\ &\leq \frac{b}{x_i} \left( \sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right) + \frac{1}{x_i} M_i, \end{aligned} \tag{2.1}$$

since  $a_{ik}d_k/d_i \leq b$ . Similarly for  $\ell \leq j \leq n$  we have

$$\begin{aligned} r_j &= \sum_{k=1}^{\ell-1} x_k a_{jk} \frac{d_k}{d_j} + \sum_{k=\ell}^n a_{jk} \frac{d_k}{d_j} \\ &= \sum_{k=1}^{\ell-1} (x_k - 1) a_{jk} \frac{d_k}{d_j} + \sum_{k=1}^n a_{jk} \frac{d_k}{d_j} \\ &\leq b \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) + M_\ell. \end{aligned} \tag{2.2}$$

Let

$$\phi_\ell = \frac{M_\ell - b + \sqrt{(M_\ell + b)^2 + 4b \sum_{i=1}^{\ell-1} (M_i - M_\ell)}}{2}.$$

For  $1 \leq i \leq \ell - 1$  let

$$x_i = 1 + \frac{M_i - M_\ell}{\phi_\ell + b} \geq 1. \tag{2.3}$$

Then for  $1 \leq i \leq \ell - 1$  we have

$$\begin{aligned} r_i &\leq \frac{b}{x_i} \left( \sum_{k=1, k \neq i}^{\ell-1} x_k - (\ell - 2) \right) + \frac{1}{x_i} M_i \\ &= \frac{b \sum_{k=1}^{\ell-1} (M_k - M_\ell) + \phi_\ell M_i + b M_\ell}{\phi_\ell + b + M_i - M_\ell} \\ &= \frac{\frac{1}{4}[(M_\ell - b)^2 + (M_\ell + b)^2 + 4b \sum_{k=1}^{\ell-1} (M_k - M_\ell) - 2M_\ell^2 - 2b^2 + 4bM_\ell] + \phi_\ell M_i}{\phi_\ell + b + M_i - M_\ell} \end{aligned}$$

$$\begin{aligned} &= \frac{\phi_\ell^2 + \phi_\ell b - \phi_\ell M_\ell + \phi_\ell M_i}{\phi_\ell + b + M_i - M_\ell} \\ &= \phi_\ell. \end{aligned}$$

For  $\ell \leq j \leq n$  we have

$$\begin{aligned} r_i &\leq b \left( \sum_{k=1}^{\ell-1} x_k - (\ell - 1) \right) + M_\ell \\ &= \frac{b \sum_{k=1}^{\ell-1} (M_k - M_\ell) + \phi_\ell M_\ell + b M_\ell}{\phi_\ell + b} \\ &= \frac{\frac{1}{4} [4b \sum_{k=1}^{\ell-1} (M_k - M_\ell) + 2M_\ell \sqrt{(M_\ell + b)^2 + 4b \sum_{k=1}^{\ell-1} (M_k - M_\ell) + 2M_\ell^2 + 2bM_\ell}]}{\phi_\ell + b} \\ &= \frac{\phi_\ell^2 + \phi_\ell b}{\phi_\ell + b} \\ &= \phi_\ell. \end{aligned}$$

Hence by Theorem 1.1,

$$\rho(G) = \rho(B) \leq \max_{1 \leq i \leq n} \{r_i\} \leq \phi_\ell.$$

The first part of Theorem 1.4 follows.

Suppose  $M_1 = M_2 = \dots = M_n$ . Then  $\rho(G) = M_1 = \phi_\ell$  by Theorem 1.2. Hence the equality in Theorem 1.4 follows.

To prove the necessary condition, suppose  $\rho(G) = \phi_\ell$ . Applying Theorem 1.1 and the inequalities in (2.1) and (2.2),  $\phi_\ell = \rho(G) \leq \max_{1 \leq i \leq n} r_i \leq \phi_\ell$ . Hence  $r_1 = r_2 = \dots = r_n = \phi_\ell$ , and the equalities in (2.1) and (2.2) hold. In particular,

$$b = a_{ik} \frac{d_k}{d_i} \tag{2.4}$$

for any  $1 \leq i \leq n$  and  $1 \leq k \leq \ell - 1$  with  $x_k - 1 > 0$ , and  $M_\ell = M_n$ . We consider three cases:

- (1) Suppose  $M_1 = M_\ell$ : Clearly  $M_1 = M_n$ .
- (2)  $M_{t-1} > M_t = M_\ell$  for some  $3 \leq t \leq \ell$ : Then  $x_k > 1$  for  $1 \leq k \leq t - 1$  by (2.3). Hence by (2.4)

$$b = a_{12} \frac{d_2}{d_1} = a_{21} \frac{d_1}{d_2} = 1,$$

and  $d_i = n - 1$  for all  $i = 1, 2, \dots, n$ . This implies that  $G$  is regular, a contradiction.

- (3)  $M_1 > M_2 = M_\ell$ : Then  $x_1 > 1$  by (2.3). Hence by (2.4),  $b = a_{i1} d_1 / d_i$  for  $2 \leq i \leq n$ . Hence  $d_1 = n - 1$  and  $d_2 = d_3 = \dots = d_n = (n - 1) / b$ . Then  $(n - 1) / b = M_1 > M_2 = M_n = (n - 1) / b - 1 + b$ . This implies  $b < 1$ , a contradiction.

This completes the proof of the theorem. □

**Example 2.1.** The graphs in Figs. 1–3 are pseudo-regular but not regular. In particular, the graph in Fig. 3 has a cycle  $C_k$  of  $k$  vertices, and shares each vertex of  $C_k$  with a triangle  $K_3$ .

An interesting problem could be characterizing all the nonregular pseudo-regular graphs.

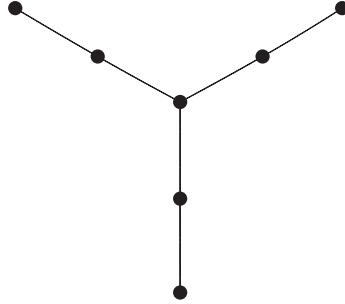


Fig. 1. Graph with  $M_i = 2$ .

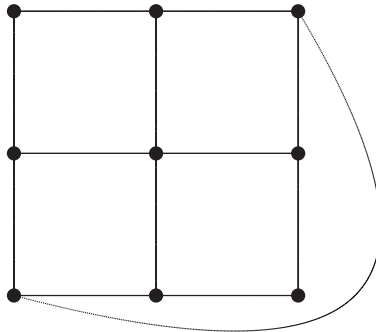


Fig. 2. Graph with  $M_i = 3$ .

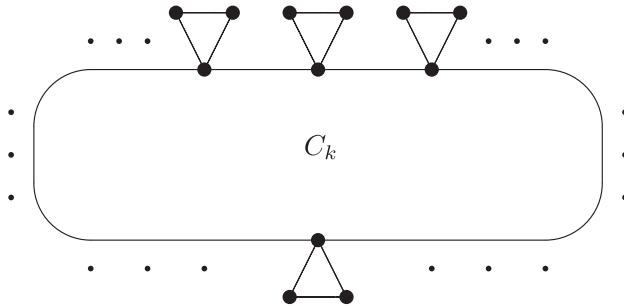


Fig. 3. Graphs with  $M_i = 3$ .

**3. The Shape of the Sequence  $\phi_1, \phi_2, \dots, \phi_n$**

Given a decreasing sequence  $M_1 \geq M_2 \geq \dots \geq M_n$  of positive integers, consider the functions

$$\phi_\ell(x) = \frac{M_\ell - x + \sqrt{(M_\ell + x)^2 + 4x \sum_{i=1}^{\ell-1} (M_i - M_\ell)}}{2}$$

for  $x \in [1, \infty)$ . Note that  $\phi_\ell(b)$  is the upper bound of  $\rho(G)$  in Theorem 1.4.

The following proposition shows that the smaller the  $b$  in Theorem 1.4 is, the lower the upper bound of  $\rho(G)$  reaches.

**Proposition 3.1.** *For any  $1 \leq \ell \leq n$ ,  $\phi_\ell(x)$  is increasing on  $[1, \infty)$ .*

**Proof.** For convenience, let

$$S = \sum_{i=1}^{\ell-1} (M_i - M_\ell).$$

To show that  $\phi_\ell(x)$  is increasing on  $[1, \infty)$ , it is sufficient to show that the derivative of  $\phi_\ell(x)$  is nonnegative. This follows from the following equivalent steps.

$$\begin{aligned} \phi'_\ell(x) &\geq 0 \\ \Leftrightarrow -1 + \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} &\geq 0 \\ \Leftrightarrow \frac{M_\ell + x + 2S}{\sqrt{(M_\ell + x)^2 + 4Sx}} &\geq 1 \\ \Leftrightarrow (M_\ell + x + 2S)^2 &\geq (M_\ell + x)^2 + 4Sx \\ \Leftrightarrow 4SM_\ell + 4S^2 &\geq 0. \quad \square \end{aligned}$$

Note that for  $1 \leq s \leq n - 1$ ,  $M_s = M_{s+1}$  implies  $\phi_s(x) = \phi_{s+1}(x)$ . We adopt the same viewpoint as [9, Proposition 3.1] to describe when the bound gets improved throughout the sequence  $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$  in the following proposition.

**Proposition 3.2.** *Suppose  $M_s > M_{s+1}$  for some  $1 \leq s \leq n - 1$ , and let the symbol  $\succeq$  denote  $>$  or  $=$ . Then*

$$\phi_s(x) \succeq \phi_{s+1}(x) \quad \text{if and only if} \quad \sum_{i=1}^s M_i \succeq xs(s - 1).$$

**Proof.** Consider the following equivalent relations step by step.

$$\begin{aligned} \phi_s(x) &> \phi_{s+1}(x) \\ \Leftrightarrow M_s - M_{s+1} + \sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} &> M_{s+1} - M_{s+2} + \sqrt{(M_{s+1} + x)^2 + 4x \sum_{i=1}^s (M_i - M_{s+1})} \end{aligned}$$

$$\begin{aligned}
 &> \sqrt{(M_{s+1} + x)^2 + 4x \sum_{i=1}^s (M_i - M_{s+1})} \\
 \Leftrightarrow &\sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} > 2xs - (M_s + x) \\
 \Leftrightarrow &(M_s + x)^2 + 4x \sum_{i=1}^s (M_i - M_s) > 4x^2s^2 - 4xs(M_s + x) + (M_s + x)^2 \\
 \Leftrightarrow &\sum_{i=1}^s M_i > xs(s - 1),
 \end{aligned}$$

where the third relation is obtained from the second by taking square on both sides, simplifying it, and deleting the common term  $M_s - M_{s+1}$ . Note that even if  $2xs - (M_s + x) < 0$  in the third relation, squaring both sides would be proper since then  $\sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} \geq |M_s + x| \geq |2xs - (M_s + x)|$ . Similarly, note that if  $\sum_{i=1}^s M_i = xs(s - 1)$ , then  $M_s \leq xs$  and  $2xs - (M_s + x) \geq 0$ . Hence

$$\phi_s(x) = \phi_{s+1}(x) \tag{3.1}$$

$$\begin{aligned}
 \Leftrightarrow &\sqrt{(M_s + x)^2 + 4x \sum_{i=1}^{s-1} (M_i - M_s)} = 2xs - (M_s + x) \\
 \Leftrightarrow &(M_s + x)^2 + 4x \sum_{i=1}^s (M_i - M_s) = 4x^2s^2 - 4xs(M_s + x) + (M_s + x)^2 \\
 \Leftrightarrow &\sum_{i=1}^s M_i = xs(s - 1). \quad \square
 \end{aligned}$$

The following corollary shows that Theorem 1.3 is an improvement of Theorem 1.2.

**Corollary 3.3.** *For any  $x \in [1, \infty)$ ,  $\phi_2(x) \leq M_1$  with equality if and only if  $M_2 = M_1$ .*

**Proof.** If  $M_2 = M_1$  then  $\phi_2(x) = M_2 \leq M_1$ . Suppose  $M_2 < M_1$ . Choose  $s = 1$  and the symbol  $\succeq$  to be  $>$  in Proposition 3.2,

$$M_1 = \phi_1(x) > \phi_2(x). \quad \square$$

Choosing  $b = \max\{d_i/d_j \mid i \sim j\}$ , by Proposition 3.2 with  $s = 2$  and  $x = b$ , if  $M_2 > M_3$  and  $M_1 + M_2 > 2b$ , then  $\phi_2(b) > \phi_3(b)$ . This is a case when Theorem 1.4 is truly an improvement of Theorem 1.3.

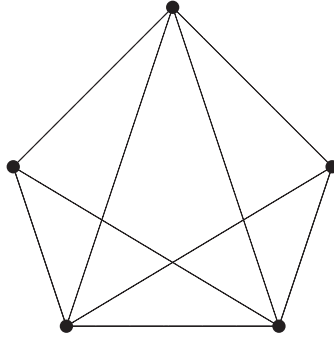


Fig. 4. Graph with  $\phi_2 > \phi_3$ .

**Example 3.4.** For the graph in Fig. 4,  $M_1 = M_2 = 4$ ,  $M_3 = 7/2$ ,  $b = 4/3$ ,  $\phi_1(b) = \phi_2(b) = 4$ ,  $\phi_3(b) \doteq 3.762$  and  $\rho(G) = 1 + \sqrt{7} \doteq 3.646$ .

Note that  $\phi_1(x) = M_1 \geq \phi_2(x)$  by Corollary 3.3, and for  $2 \leq t \leq n - 1$ ,  $\sum_{i=1}^t M_i < xt(t - 1)$  implies  $M_t < x(t - 1)$ , and hence  $\sum_{i=1}^{t+1} M_i < xt(t - 1) + x(t - 1) < xt(t + 1)$ . This implies that the sequence  $\phi_1(x), \phi_2(x), \dots, \phi_n(x)$  is composed by two parts. The first part is decreasing and the second part is increasing. In particular, if we choose  $x = M_1$ ,  $M_2 > M_3$ ,  $s = 2$  and  $\succeq$  to be  $>$  in Proposition 3.2, then  $M_1 + M_2 \not\geq 2M_1 = xs(s - 1)$ , so  $\phi_2(M_1) \leq \phi_3(M_1)$ . Hence  $\phi_2(M_1)$  is smallest among  $\phi_1(M_1), \phi_2(M_1), \dots, \phi_n(M_1)$ .

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