A bound on the Laplacian spread which is tight for strongly regular graphs

Fan-Hsuan Lin[a,*], Chih-wen Weng[a]

[a]*Department of Applied Mathematics, National Chiao Tung University, 1001 Ta Hsueh Road, Hsinchu, Taiwan, ROC.*

Abstract

The Laplacian spread of a graph is the difference between the largest eigenvalue and the second-smallest eigenvalue of the Laplacian matrix of the graph. For Laplacian matrices of graphs, we find their upper bounds of largest eigenvalues, lower bounds of second-smallest eigenvalues and upper bounds of Laplacian spreads. The strongly regular graphs attain all the above bounds. Some other extremal graphs are also provided.

Keywords: Laplacian matrix, Laplacian spread, strongly regular graph.

1. Introduction

Let $G = (V, E)$ be a simple connected graph of order n with vertex set $V = \{1, 2, \cdots, n\}$ and edge set E. Let $A = A(G)$ be the *adjacency matrix* of G, i.e. the binary matrix with ij-entry 1 if and only if i and j are distinct and adjacent. The *degree* d_i of vertex $i \in V$ is the number $|N(i)|$, where $N(i)$ is the set of vertices which are adjacent to i. Let $D(G)$=diag$(d_1, d_2, \cdots, d_n)$ be the diagonal matrix with entries $d_1, d_2, \cdots, d_n$ in the diagonal. Then the matrix

$$L(G) = D(G) - A(G)$$

is called the *Laplacian matrix* of G. We call the eigenvalues of $L(G)$ the *Laplacian eigenvalues* of G. It is well-known that $L(G)$ is symmetric, positive semidefinite and every row-sum being zero [8], so we denote the Laplacian eigenvalues of G in nonincreasing order as $\ell_1(G) \geq \ell_2(G) \geq \cdots \geq \ell_n(G) = 0$. The eigenvalues $\ell_1(G)$, $\ell_{n-1}(G)$ and $\ell_n(G)$ are called *Laplacian index*, *algebraic connectivity* and *trivial eigenvalue*, respectively. The *Laplacian spread* of G is defined as $\mathscr{S}_L(G) := \ell_1(G) - \ell_{n-1}(G)$.

A graph G is called *k-regular* if every vertex of G has degree k. Moreover, G is called *strongly regular* with parameters (n, k, λ, μ) if G is a k-regular graph with order n which has λ (resp. μ) common neighbors of any pair of two adjacent (resp. nonadjacent) vertices. The Laplacian matrix of a k-regular graph is $kI - A$, so its eigenvalues are easily obtained from those of the adjacency matrix. It is well-known that a strongly regular graph with parameters (n, k, λ, μ) and $n \neq k + 1$ has two distinct nontrivial Laplacian eigenvalues $\ell_1(G), \ell_{n-1}(G)$, and indeed

$$\ell_1(G), \ell_{n-1}(G) = \frac{2k - \lambda + \mu \pm \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}}{2}. \tag{1.1}$$

See for example [11, Chapter 21]. Note that G is strongly regular if and only if its complement G^c is strongly regular [2, Theorem 1.3.1].

Let $\delta := \min_{i \in V} d_i$ and $\Delta := \max_{i \in V} d_i$. Motivated by the definition of strongly regular graphs, we define another two graph parameters $\lambda(G)$ and $\mu(G)$ for any graph G :

$$\lambda(G) := \min_{ij \in E} |N(i) \cap N(j)|, \qquad \mu(G) := \min_{ij \notin E} |N(i) \cap N(j)|.$$

*Corresponding author

Email addresses: `fanhsuan.am03g@nctu.edu.tw` (Fan-Hsuan Lin), `weng@math.nctu.edu.tw` (Chih-wen Weng)

In Section 2, we review some previously known bounds for the Laplacian index $\ell_1(G)$ and algebraic connectivity $\ell_{n-1}(G)$. Our main theorem, Theorem 3.1 in Section 3, is an inequality involving the graph parameters n, d_i, m_i, $\lambda(G)$, $\mu(G)$ and a nontrivial eigenvalue ℓ with its corresponding eigenvector $(x_1, x_2, \ldots, x_n)^T$ of a graph G, where $m_i = (\sum_{ji\in E} d_j)/d_i$ is called the *average* 2*-degree* of a vertex i. We give two ways to eliminate the eigenvector parameters in the above inequality, and provide two lower bounds and two upper bounds of the nontrivial Laplacian eigenvalue ℓ. Among these, the upper bounds for $\ell = \ell_1(G)$ and lower bounds of $\ell = \ell_{n-1}(G)$ are of interest. See Corollary 3.2, and (3.12), (3.13) in Corollary 3.3. We find that the class of strongly regular graphs attains both bounds. In Section 4, we provide more extremal graphs, attaining bounds of some, but not all of the inequalities in Section 3. In Section 5, the Laplacian index $\ell_1(G)$ and algebraic connectivity $\ell_{n-1}(G)$ for all connected $(n-3)$-regular graphs G of order n are determined. With a class of exceptions, they are extremal for all the inequalities obtained in this paper.

2. Some known bounds

We recall some basic properties of Laplacian matrices [15] and known bounds about Laplacian index $\ell_1(G)$ and algebraic connectivity $\ell_{n-1}(G)$ in this section. One will find the basic properties of them in [15].

Perhaps the most important property of $L(G)$ is

$$X^\top L(G)X = \sum_{j<k,\, jk\in E} (x_j - x_k)^2, \tag{2.1}$$

where X is a column vector. As the property $L(G) + L(G^c) = nI - J$, where J is the all-ones matrix, the Laplacian matrices $L(G)$ and $L(G^c)$ of G and its complement G^c respectively share the same set of eigenvectors. Hence if ℓ is a nontrivial eigenvalue of $L(G)$ with associated eigenvector X, then $n - \ell$ is an eigenvalue of $L(G^c)$ with the same associated eigenvector X. Thus a bound of $\ell_1(G^c)$ also gives a bound of $\ell_{n-1}(G)$.

In 1973 [8], Fiedler showed the following upper bound of $\ell_{n-1}(G)$

$$\ell_{n-1}(G) \le \kappa(G) \le \delta, \tag{2.2}$$

stimulating the study of algebraic connectivity, where $\kappa(G)$ is the vertex connectivity of G. Note that $\ell_{n-1}(G) = 0$ if and only if G is disconnected.

The upper bounds of Laplacian index $\ell_1(G)$ were studied by many authors. In 1985 [1], Anderson and Morley showed that

$$\ell_1(G) \le \max_{ij\in E} \left\{d_i + d_j\right\}. \tag{2.3}$$

Note that $d_i + m_i = d_i + (\sum_{ji\in E} d_j)/d_i \le d_i + \max_{ji\in E}\{d_j\} \le \max_{ji\in E}\{d_i + d_j\}$. In 1998 [14], Merris improved the bound in (2.3) by showing

$$\ell_1(G) \le \max_{i\in V} \{d_i + m_i\}. \tag{2.4}$$

As another way to improve the bound in (2.3), in 2000 [16], Rojo et al. showed

$$\ell_1(G) \le \max_{ij\in E} \left\{d_i + d_j - |N(i) \cap N(j)|\right\}. \tag{2.5}$$

In 2001 [12], Li and Pan gave the following bound

$$\ell_1(G) \le \max_{i\in V} \left\{ \sqrt{2d_i(d_i + m_i)}\right\}. \tag{2.6}$$

In 2004 [20], Zhang showed the following result, which is always better than the bound (2.6).

$$\ell_1(G) \le \max_{i\in V} \left\{d_i + \sqrt{d_i m_i}\right\}. \tag{2.7}$$

One of our results in Corollary 3.2 is an extension of (2.7).

For the lower bound of $\ell_1(G)$ in 1994 [9], Grone and Merris showed that

$$\ell_1(G) \ge \Delta + 1. \tag{2.8}$$

The studies of Laplacian spread $\mathscr{S}_L(G) = \ell_1(G) - \ell_{n-1}(G)$ will be found in [3, 5, 6, 7, 19]. They are interested in graphs with a few more edges than the number of edges in a tree. The results in this paper are instead concerned with graphs of higher vertex connectivity.

3. New bounds

The following is our main theorem.

Theorem 3.1. *Let $G = (V, E)$ be a simple connected graph of order n. Let ℓ be a nontrivial Laplacian eigenvalue of G with associated eigenvector $X = (x_1, x_2, \cdots, x_n)^\top$. Let d_i and m_i be degree and average 2-degree respectively of vertex $i \in V$, and let $\lambda \leq \lambda(G)$ and $\mu \leq \mu(G)$ be two given numbers. Then*

$$\sum_{i=1}^{n}[(d_i - \ell)^2 - d_i m_i + \lambda\ell + \mu(n - \ell)]x_i^2 \leq 0. \tag{3.1}$$

Moreover, the equality in (3.1) holds if and only if $\lambda = \lambda(G)$, $\mu = \mu(G)$ and for any distinct vertices $i, j \in V$, the following two statements hold:

$$ij \in E \quad \text{and} \quad x_i \neq x_j \quad \Rightarrow \quad |N(i) \cap N(j)| = \lambda(G), \tag{3.2}$$

$$ij \notin E \quad \text{and} \quad x_i \neq x_j \quad \Rightarrow \quad |N(i) \cap N(j)| = \mu(G). \tag{3.3}$$

Proof. Because X is an eigenvector of $L(G)$ corresponding to ℓ,

$$\|(D(G) - \ell I)X\|^2 = \|(D(G) - L(G))X\|^2 = \|A(G)X\|^2. \tag{3.4}$$

As the ij-entry of $A(G)^2$ is the number w_{ij} of walks of length 2 from i to j and noting that $w_{ii} = d_i$, we have

$$\begin{aligned}
\|A(G)X\|^2 &= X^\top A(G)^2 X \qquad (3.5)\\
&= \sum_{i\in V} d_i x_i^2 + 2\sum_{j<k} w_{jk}x_j x_k\\
&= \sum_{i\in V} d_i x_i^2 + \sum_{j<k} w_{jk}(x_j^2 + x_k^2 - (x_j - x_k)^2)\\
&= \sum_{i\in V}((\sum_{ji\in E} d_j)x_i^2) - \sum_{\substack{j<k\\ jk\in E}} w_{jk}(x_j - x_k)^2 - \sum_{\substack{j<k\\ jk\notin E}} w_{jk}(x_j - x_k)^2. \qquad (3.6)
\end{aligned}$$

As $\lambda \leq \lambda(G) = \min_{ij\in E} w_{ij}$ and $\mu \leq \mu(G) = \min_{ij\notin E} w_{ij}$ and by (3.4), (3.6), we have

$$\sum_{i\in V}(d_i - \ell)^2 x_i^2 \leq \sum_{i\in V} d_i m_i x_i^2 - \lambda \sum_{\substack{j<k\\ jk\in E}}(x_j - x_k)^2 - \mu \sum_{\substack{j<k\\ jk\notin E}}(x_j - x_k)^2. \tag{3.7}$$

Applying (2.1) and that $n - \ell(G)$ is eigenvalue of $L(G^c)$ with the same eigenvector X to (3.7), we have

$$\begin{aligned}
\sum_{i\in V}(d_i - \ell)^2 x_i^2 &\leq \sum_{i\in V} d_i m_i x_i^2 - \lambda X^\top L(G)X - \mu X^\top L(G^c)X\\
&= \sum_{i\in V} d_i m_i x_i^2 - \lambda\ell\|X\|^2 - \mu(n - \ell)\|X\|^2\\
&= \sum_{i\in V} d_i m_i x_i^2 - \lambda\ell \sum_{i\in V} x_i^2 - \mu(n - \ell)\sum_{i=1}^{n} x_i^2,
\end{aligned}$$

and (3.1) immediately follows from this. Note that the equality holds in (3.1) if and only if the equality in (3.7) holds, and this is equivalent to (3.2), (3.3). □

The expression

$$(d_i - \ell)^2 - d_i m_i + \lambda\ell + \mu(n - \ell) = \ell^2 - (2d_i - \lambda + \mu)\ell + (d_i^2 - d_i m_i + \mu n) \tag{3.8}$$

inside the summation in (3.1) is a quadratic polynomial in variable ℓ and is not positive for some i. Solving the quadratic polynomial, we have the following upper bound of Laplacian index $\ell_1(G)$, lower bound of the algebraic connectivity $\ell_{n-1}(G)$ and upper bound of Laplacian spread $\mathscr{S}_L(G)$ of G.

Corollary 3.2. *Referring to the notations in Theorem 3.1, the following three inequalities hold:*

$$\ell_1(G) \leq \max_{i\in V}\left\{\frac{2d_i-\lambda+\mu+\sqrt{4d_im_i-4(\lambda-\mu)d_i+(\lambda-\mu)^2-4\mu n}}{2}\right\}, \tag{3.9}$$

$$\ell_{n-1}(G) \geq \min_{i\in V}\left\{\frac{2d_i-\lambda+\mu-\sqrt{4d_im_i-4(\lambda-\mu)d_i+(\lambda-\mu)^2-4\mu n}}{2}\right\} \tag{3.10}$$

and

$$\begin{aligned}\mathscr{S}_L(G) \leq &\max_{i\in V}\left\{\frac{2d_i-\lambda+\mu+\sqrt{4d_im_i-4(\lambda-\mu)d_i+(\lambda-\mu)^2-4\mu n}}{2}\right\}\\ &-\min_{i\in V}\left\{\frac{2d_i-\lambda+\mu-\sqrt{4d_im_i-4(\lambda-\mu)d_i+(\lambda-\mu)^2-4\mu n}}{2}\right\},\end{aligned} \tag{3.11}$$

where the max *and* min *are over vertices $i \in V$ and exclude the terms with a negative term in the square root.* □

Note that the above upper bound of $\ell_1(G)$ with $\lambda = 0$ and $\mu = 0$ is (2.7), and with $\lambda = \lambda(G)$ and $\mu = 0$ is previously given in [10, Theorem 3.2].

The following corollary is another application of (3.1).

Corollary 3.3. *Referring to the notations in Theorem 3.1, the following three inequalities hold:*

$$\ell_1(G) \leq \frac{2\Delta-\lambda+\mu+\sqrt{(2\Delta-\lambda+\mu)^2-4\mu n}}{2}, \tag{3.12}$$

$$\ell_{n-1}(G) \geq \frac{2\delta-\lambda+\mu-\sqrt{(2\delta-\lambda+\mu)^2-4\mu n-4\delta^2+4\Delta^2}}{2}, \tag{3.13}$$

and

$$\mathscr{S}_L(G) \leq \Delta-\delta+\frac{1}{2}\left[\sqrt{(2\Delta-\lambda+\mu)^2-4\mu n}+\sqrt{(2\delta-\lambda+\mu)^2-4\mu n-4\delta^2+4\Delta^2}\right], \tag{3.14}$$

where δ and Δ are the maximum degree and the minimum degree in G. Moreover, if G is k-regular then

$$\mathscr{S}_L(G) \leq \sqrt{(2k-\lambda+\mu)^2-4\mu n}. \tag{3.15}$$

Proof. By (3.1) with $\ell = \ell_1(G)$ there exists $i \in V$ such that the term in (3.8) is not positive. Using $\ell_1(G) \geq \Delta + 1 > d_i$, and $\Delta \geq m_i$, we have

$$(\Delta-\ell_1(G))^2-\Delta^2+\lambda\ell_1(G)+\mu(n-\ell_1(G)) \leq (d_i-\ell_1(G))^2-d_im_i+\lambda\ell_1(G)+\mu(n-\ell_1(G)) \leq 0.$$

Solving the above quadratic inequality on the left for $\ell_1(G)$, we have (3.12). Similarly, by considering $\ell = \ell_{n-1}(G)$ in (3.1), there exists $j \in V$ such that (3.8) with $i = j$ is not positive. Using $\ell_{n-1}(G) \leq \delta \leq d_j$ and $\Delta \geq m_i$, we have

$$(\delta-\ell_{n-1}(G))^2-\Delta^2+\lambda\ell_{n-1}(G)+\mu(n-\ell_{n-1}(G)) \leq (d_j-\ell_{n-1}(G))^2-d_jm_j+\lambda\ell_{n-1}(G)+\mu(n-\ell_{n-1}(G)) \leq 0.$$

Solving the quadratic inequality on the left for $\ell_{n-1}(G)$, we have (3.13). The line (3.14) is immediate from (3.12), (3.13), and (3.15) is from (3.14). □

Next we prove that the strongly regular graphs satisfy all the above with equality.

Corollary 3.4. *If G is a strongly regular graph with parameters $(n, k, \lambda(G), \mu(G))$, then $k = \delta = \Delta$ and the equality in (3.1), the three equalities in Corollary 3.2 and the three equalities (3.9)~(3.14) all hold for $\lambda = \lambda(G)$ and $\mu = \mu(G)$.*

Proof. This is clear since (3.2) and (3.3) hold in a strongly regular graph. □

An upper bound of the largest eigenvalue of the adjacency matrix of a graph is given in [17, Theorem 1]. It has a pattern similar to (3.13), but the parameters δ, $\lambda(G)$, $\mu(G)$ are defined in a way opposite to ours. See [17] for details.

4. Other extremal graphs

A graph is *extremal* for an inequality holding for graphs if the graph attains the equality. Although we know that strongly regular graphs are extremal for any inequalities in the previous section, it seems much more difficult to classify all the extremal graphs. In this section, we shall provide some extremal graphs found by computer search, excluding strongly regular graphs. Throughout this section we assume $\lambda = \lambda(G)$ and $\mu = \mu(G)$. Let

$$\alpha_1(G) = \max_{i \in V} \left\{ \frac{2d_i - \lambda + \mu + \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\}$$

$$\left(\text{resp. } \beta_1(G) = \min_{i \in V} \left\{ \frac{2d_i - \lambda + \mu - \sqrt{4d_i m_i - 4(\lambda - \mu)d_i + (\lambda - \mu)^2 - 4\mu n}}{2} \right\} \right)$$

denote the upper bound (resp. lower bound) of Laplacian index $\ell_1(G)$ (resp. algebraic connectivity $\ell_{n-1}(G)$) described in (3.9) (resp. (3.10)) and let

$$\alpha_2(G) = \frac{2\Delta - \lambda + \mu + \sqrt{(2\Delta - \lambda + \mu)^2 - 4\mu n}}{2}$$

$$\left(\text{resp. } \beta_2(G) = \frac{2\delta - \lambda + \mu - \sqrt{(2\delta - \lambda + \mu)^2 - 4\mu n - 4\delta^2 + 4\Delta^2}}{2} \right)$$

denote the upper bound (resp. lower bound) of Laplacian index $\ell_1(G)$ (resp. algebraic connectivity $\ell_{n-1}(G)$) described in (3.12) (resp. (3.13)). In Example 4.1 and Example 4.2, we provide some regular graphs which are extremal for all mentioned inequalities (3.9)~(3.14).

Example 4.1. The graph $G = X_8$ depicted on the left of Figure 1 is 3-regular of order 8 with $\lambda(X_8) = 0, \mu(X_8) = 1$ and $\ell_1(X_8),\ \ell_7(X_8) = (7 \pm \sqrt{17})/2 = \ell,\ \mathscr{S}_L(G) = \sqrt{17}$, so it is extremal for (3.1). Note that $\alpha_1(X_8) = \alpha_2(X_8) = (7 + \sqrt{17})/2$ and $\beta_1(X_8) = \beta_2(X_8) = (7 - \sqrt{17})/2$. Hence X_8 is extremal for the inequalities (3.9)~(3.14). On the other hand, the complement graph $G^c = X_8^c$ of X_8 has $\lambda(X_8^c) = 1, \mu(X_8^c) = 2$ and $\ell_1(X_8^c), \ell_7(X_8^c) = (9 \pm \sqrt{17})/2 = \ell, \mathscr{S}_L(X_8^c) = \sqrt{17}$, so it is also extremal for (3.1). Note that $\alpha_1(X_8^c) = \alpha_2(X_8^c) = (9 + \sqrt{17})/2$ and $\beta_1(X_8^c) = \beta_2(X_8^c) = (9 - \sqrt{17})/2$. Hence X_8^c is extremal for the inequalities (3.9)~(3.14).

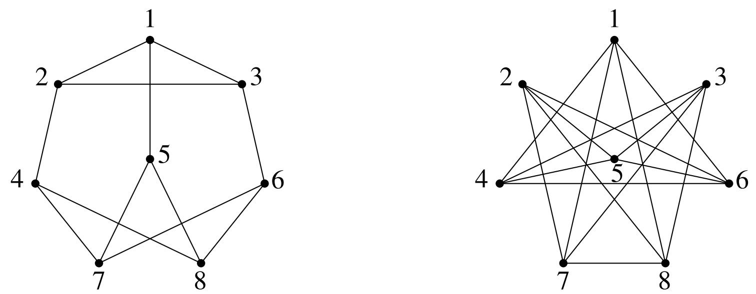

Figure 1: The graph X_8 on the left has $\lambda(X_8) = 0, \mu(X_8) = 1$ and its complement graph X_8^c on the right has $\lambda(X_8^c) = 1, \mu(X_8^c) = 2$.

Example 4.2. The graph $G = Y_8$ depicted on the left of Figure 2 is obtained from the complete graph K_8 of order 8 by deleting two vertex disjoint cycles of order 4. It is 5-regular of order 8 with $\lambda(Y_8) = 2$, $\mu(Y_8) = 4$, $\ell_1(Y_8) = 8$ and $\ell_7(Y_8) = 4$, so it is extremal for (3.1) with $\ell = 8,\ 4$, respectively. Note that $\alpha_1(Y_8) = \alpha_2(Y_8) = 8, \beta_1(Y_8) = \beta_2(Y_8) = 4$. Hence G is extremal for the inequalities (3.9)~(3.14).

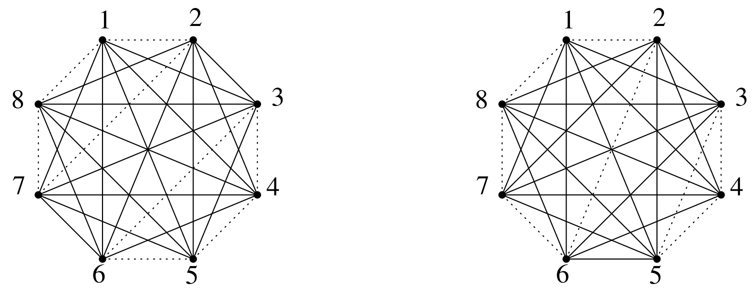

Figure 2: The graph Y_8 on the left has $\lambda(Y_8) = 2$, $\mu(Y_8) = 4$, and the graph Z_8 on the right has $\lambda(Z_8) = 2$, $\mu(Z_8) = 4$.

In Example 4.3 and Example 4.4, we provide some regular graphs which are extremal only for all inequalities about $\ell_1(G)$ and some regular graphs which are extremal only for all inequalities about $\ell_{n-1}(G)$.

Example 4.3. The graph $G = Z_8$ depicted on the right of Figure 2 is obtained from K_8 by deleting two vertex disjoint cycles of order 3 and 5, respectively. It is 5-regular of order 8 with $\lambda(Z_8) = 2$, $\mu(Z_8) = 4$, $\ell_1(Z_8) = 8$, so it is extremal for (3.1) with $\ell = 8$. Note that $\alpha_1(Z_8) = \alpha_2(Z_8) = 8$, $\beta_1(Z_8) = \beta_2(Z_8) = 4$ and $\ell_7(Z_8) = (11 - \sqrt{5})/2$. Hence Z_8 is extremal for the inequalities (3.9) and (3.12) and is not extremal for other inequalities.

Example 4.4. The graph $G = U_8$ depicted on the left of Figure 3 is 4-regular of order 8 with $\lambda(U_8) = 0$, $\mu(U_8) = 2$, $\ell_7(U_8) = 2$, so it is extremal for (3.1) with $\ell = 2$. Note that $\alpha_1(U_8) = \alpha_2(U_8) = 8$, $\beta_1(U_8) = \beta_2(U_8) = 2$ and $\ell_1(U_8) = 6$. Hence U_8 is extremal for the inequalities (3.10) and (3.13), and is not extremal for other inequalities. On the other hand, the complement graph $G = U_8^c$ of U_8 depicted on the right of Figure 3 is 3-regular of order 8 with $\lambda(U_8^c) = \mu(U_8^c) = 0$, $\ell_1(U_8^c) = 6$, so it is extremal for (3.1) with $\ell = 6$. Note that $\alpha_1(U_8^c) = \alpha_2(U_8^c) = 6$, $\beta_1(U_8^c) = \beta_2(U_8^c) = 0$ and $\ell_7(U_8^c) = 2$. Hence U_8^c is extremal for the inequalities (3.9) and (3.12), and is not extremal for other inequalities.

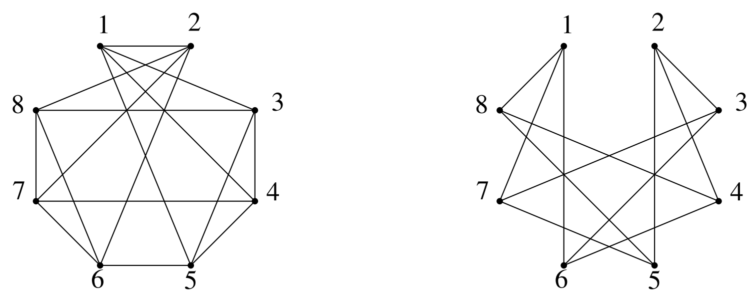

Figure 3: The graph U_8 on the left has $\lambda(U_8) = 0$, $\mu(U_8) = 2$, and its complement graph U_8^c on the right has $\lambda(U_8^c) = \mu(U_8^c) = 0$.

All the above extremal graphs are regular. We conjecture that only regular graphs are extremal for the inequalities (3.12), (3.13) or (3.14). However an eigenvector also helps to obtain the equality in (3.1), we do find nonregular graphs which are extremal for this inequality. The inequality (3.10) does not have eigenvector involved, but we also find nonregular graphs which are extremal for (3.10) in the following two examples.

Example 4.5. Let $G = K_{a,b}$ be the complete bipartite graph of bipartition orders a and b, respectively, where $a < b$ and $n = a + b$. Then $\lambda(K_{a,b}) = 0$, $\mu(K_{a,b}) = a$, $(d_i, m_i) = (a, b)$ or (b, a), so the lower bound of $\ell_{n-1}(K_{a,b})$ in the second inequality of Corollary 3.2 is $\beta_1(K_{a,b}) = \min\{a,\ (2b + a - \sqrt{a(4b - 3a)})/2\} = a$. Also $\ell_{n-1}(K_{a,b}) = a$ since

$\ell_{n-1}(K_{a,b}) \le \kappa(K_{a,b}) \le \delta = a$. Hence $K_{a,b}$ is extremal for (3.1) with $\ell = \ell_{n-1}(K_{a,b})$ and the inequality (3.10). Note that $\alpha_1(K_{a,b}) = (2b + a + \sqrt{a(4b-3a)})/2$, $\alpha_2(K_{a,b}) = (2b + a + \sqrt{4b^2 - 3a^2})/2$, $\beta_2(K_{a,b}) = 2a - b$ and $X = (b\mathbf{1}_{2t}^\top, -a\mathbf{1}_{2t}^\top)^\top$ is an eigenvector corresponding to the eigenvalue $\ell_1(K_{a,b}) = a + b$, where $\mathbf{1}$ is all-ones vector. Hence one can check that $K_{a,b}$ is extremal for (3.1) with $\ell = \ell_1(K_{a,b})$ and $K_{a,b}$ is not extremal for other inequalities.

Example 4.6. Let $G = F_t$ $(t > 1)$ be a friendship graph of order $n = 2t + 1$ as depicted in Figure 4. Then $\lambda(F_t) = 1$, $\mu(F_t) = 1$, $(d_i, m_i) = (2, t + 1)$ or $(2t, 2)$, and $X = (\mathbf{1}_{2t}^\top, -2t)^\top$ is an eigenvector corresponding to the eigenvalue $\ell_1(F_t) = 2t+1$. One can check that F_t is extremal for (3.1) with $\ell = \ell_1(F_t)$. On the other hand, $\alpha_1(F_t) = (2t + \sqrt{2t-1})$, $\alpha_2(F_t) = 2t + \sqrt{4t^2 - 2t - 1}$, $\beta_1(F_t) = \min\{2t - \sqrt{2t-1},\ 1\} = 1 \le \ell_{2t}(F_t) \le \kappa(F_t) = 1$, $\beta_2(F_t) = 2 - \sqrt{4t^2 - 2t - 1}$, so F_t is also extremal for (3.1) with $\ell = \ell_{2t}(F_t)$, the inequality (3.10), but is not extremal for any other inequalities.

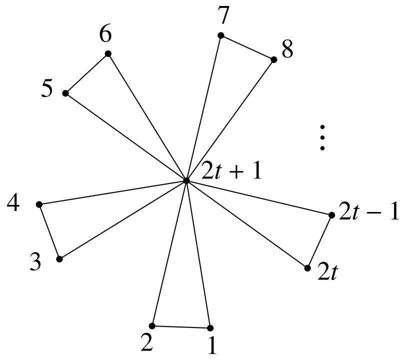

Figure 4: The friendship graph F_t of order $2t + 1$ with $\lambda(F_t) = 1$, $\mu(F_t) = 1$.

Every graph in Example 4.5 and Example 4.6 is extremal only for (3.1) about $\ell_1(G)$ because every vertex i in the graph does not satisfy $(d_i - \ell_1(G))^2 - d_i m_i + \lambda\ell_1(G) + \mu(n - \ell_1(G)) = 0$.

The following table summarizes the extremal graphs mentioned in this section which are not strongly regular.

Graph	(3.1) with $\ell = \ell_1(G)$	$\ell_1(G) = \alpha_1(G)$	$\ell_1(G) = \alpha_2(G)$	(3.1) with $\ell = \ell_{n-1}(G)$	$\ell_{n-1}(G) = \beta_1(G)$	$\ell_{n-1}(G) = \beta_2(G)$
X_8	√	√	√	√	√	√
X_8^c	√	√	√	√	√	√
Y_8	√	√	√	√	√	√
Z_8	√	√	√	×	×	×
U_8	×	×	×	√	√	√
U_8^c	√	√	√	×	×	×
$K_{a,b}$	√	×	×	√	√	×
F_t	√	×	×	√	√	×

Table 1: Some extremal graphs that are not strongly regular (√ ="YES", × ="NO").

5. The $(n-3)$-regular graphs of order n

From now on let G denote an $(n-3)$-regular graph G of order n. Note that G is obtained from the complete graph K_n by deleting some edges whose union forms vertex disjoint cycles of order n. Denote G in notation $G = K_n - (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})$, where $n_1 \ge n_2 \ge \ldots \ge n_t \ge 3$ is a nonincreasing integer sequence satisfying $n = n_1 + n_2 + \cdots + n_t$. For example, $Y_8 = K_8 - 2C_4$ in Example 4.2 and $Z_8 = K_8 - (C_3 \cup C_5)$ in Example 4.3. Note that G is connected if and only if $n \ge 5$. If $n = 5$ then $G = K_5 - C_5 = C_5$ is a cycle of order 5 and is a strongly regular graph with $\ell_1(G)$, $\ell_4(G) = (5 \pm \sqrt{5})/2$. Hence we assume $n \ge 6$.

Proposition 5.1. *The* $(n-3)$*-regular graph* $G = K_n - (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})$ *has* $\lambda(G) = n-6$ *and*

$$\mu(G) = \begin{cases} n-3, & \textit{if } n_i = 3 \textit{ for all } i; \\ n-4, & \textit{otherwise.} \end{cases}$$

Moreover, G is strongly regular if and only if $n_i = 3$ *for all* $1 \le i \le t$*, and in this case* $\ell_1(G) = n$ *and* $\ell_{n-1}(G) = n-3$.

Proof. As G is $(n-3)$-regular, $|N(i) \cap N(j)| \ge n-6$ for $i, j \in V$. To prove $\lambda(G) = n-6$, we need to choose $ij \in E$ such that in the deleting cycles to which i, j belong, the two neighbors of i do not overlap the two neighbors of j. This is done if $n_1 \ge 6$ of course, and also is done if $n_1 < 6$ since then, as with $n \ge 6$, there are two deleting cycles and we choose i, j in different cycles.

If $i'j' \notin E$ then they are in the same deleting cycle and are adjacent in the cycle. Hence $|N(i') \cap N(j')| \ge n-4$, where the excluding four vertices are i', j', the other neighbor a of i', the other neighbor b of j', and $a = b$ if and only if i', j' are inside C_3. This proves the line of $\mu(G)$.

The only possibility for the graph G to be strongly regular is when every deleting cycle is a triangle. Hence G is strongly regular if and only if $n_i = 3$ for all i. The $\ell_1(G) = n$ and $\ell_{n-1}(G) = n-3$ are determined from (1.1) by using $\lambda(G) = n-6$ and $\mu(G) = n-3$. Therefore, we complete the proof. □

The Laplacian eigenvalues of a cycle is well-known [4, Section 1.4.3], indeed

$$\ell_1(C_s) = \begin{cases} 4, & s \text{ is even;} \\ 2 + 2\cos(\pi/s), & s \text{ is odd,} \end{cases} \tag{5.1}$$

and

$$\ell_{s-1}(C_s) = 2 - \cos(2\pi/s). \tag{5.2}$$

Proposition 5.2. *The* $(n-3)$*-regular graph* $G = K_n - (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})$ *with some* $n_i > 3$ *has* $\alpha_1(G) = \alpha_2(G) = n$, $\beta_1(G) = \beta_2(G) = n-4$,

$$\ell_1(G) = \begin{cases} n - \cos(2\pi/n), & t = 1; \\ n, & t \ge 2 \end{cases}$$

and

$$\ell_{n-1}(G) = \begin{cases} n - 2 - 2\cos(\pi/n_1), & n_i \textit{ is odd for all } i; \\ n-4, & \textit{otherwise.} \end{cases}$$

Proof. Applying $\lambda(G) = n-6$ and $\mu(G) = n-4$ to the definitions, one finds $\alpha_1(G) = \alpha_2(G) = n$ and $\beta_1(G) = \beta_2(G) = n-4$ immediately. From (5.2),

$$\ell_1(G) = n - \ell_{n-1}(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t}) = \begin{cases} n - \cos(2\pi/n), & t = 1; \\ n, & t \ge 2. \end{cases}$$

From (5.1),

$$\ell_{n-1}(G) = n - \ell_1(C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t}) = \begin{cases} n - 2 - 2\cos(\pi/n_1), & \text{if } n_i \text{ is odd for all } i; \\ n-4, & \text{otherwise.} \end{cases}$$

□

From Proposition 5.2, we find that if $t \ge 2$ and n_i is even for some i then the $(n-3)$-regular graph $G = K_n - (C_{n_1} \cup C_{n_2} \cup \cdots \cup C_{n_t})$ is extremal for (3.1) with $\ell = \ell_1(G),\ \ell_{n-1}(G)$, the three inequalities in Corollary 3.2, (3.12), (3.13) and (3.14).

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References

[1] W.N. Anderson, T.D. Morley, Eigenvalues of the Laplacian of a graph, *Linear Multilinear Algebra*, 18 (1985), 141-145.

[2] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989.

[3] Y.-H. Bao, Y.-Y. Tan, Y.-Z. Fan, The Laplacian spread of unicyclic graphs, *Applied Mathematics Letters*, 22(2009), 1011-1015.

[4] A.E. Brouwer and W. H. Haemers *Spectra of Graphs*, 2012: Springer-Verlag.

[5] Y.Q. Chen, L.G. Wang, The Laplacian spread of tricyclic graphs, *The Electronic Journal of Combinatorics*, 16 (2009), #*R*80.

[6] Y.-Z. Fan, S.-D. Li, Y.-Y. Tan, The Laplacian spread of bicyclic graphs, *Journal of Mathematical Research and Exposition*, Jan., vol.30 No. 1 (2010), pp. 17-28.

[7] Y.-Z. Fan, J. Xu, Y. Wang, D. Liang, The Laplacian spread of a tree, *Discrete Mathematics and Theoretical Computer Science*, vol.10:1(2008), 79-86.

[8] M. Fiedler, Praha, Algebraic connectivity of graphs, *Czechoslovak Mathematical Journal* 23 (98), (1973), Praha.

[9] R. Grone and R. Merris, The Laplacian spctrum of a graph II, *SIAM Journal Discrete Math,* 7:229-237, 1994.

[10] J.M. Guo, J. Li, W.C. Shiu, A note on the upper bounds for the Laplacian spectral radius of graphs, *Linear Algebra and its Applications*, 439 (2013), 1657-1661.

[11] J.H. van Lint and R.M. Wilson, *A course in combinatorics,* Cambridge University Press, 2001.

[12] J.S. Li, Y.L. Pan, De Caen's inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra and its Applications*, 328 (2001), 153-160.

[13] R. Merris, Laplacian graph eigenvectors, *Linear Algebra and its Applications*, 278 (1998), 221-236.

[14] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 285 (1998), 33-35.

[15] M. W. Newman, *The Laplacian Spectrum of Graphs*, University of Manitoba, Winnipeg, MB, Canada, 2000.

[16] O. Rojo, R. Soto, H. Rojo, An always nontrivial upper bound for Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 312 (2000), 155-159.

[17] L.S. Shi, Z.P. Song, Upper bounds on the spectral radius of book-free and/or $k_{2,l}$-free graphs, *Linear Algebra and its Applications*, 420 (2007), 526-529.

[18] L.S. Shi, Bounds on the (Laplacian) spectral radius of graphs, *Linear Algebra and its Applications*, 422 (2007), 755-770.

[19] Z. You, B. Liu, The minimal Laplacian spread of unicyclic graphs, *Linear Algebra and its Applications*, 432 (2010), 499-504.

[20] X.D. Zhang, Two sharp upper bounds for the Laplacian eigenvalues, *Linear Algebra and its Applications* 376 (2004), 207-213.

[21] X.D. Zhang, R. Luo, The Laplacian eigenvalues of mixed graphs, *Linear Algebra and its Applications* 362 (2003), 109-119.