

# Spectral characterizations of two families of nearly complete bipartite graphs

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It is not hard to find many complete bipartite graphs which are not determined by their spectra. We show that the graph obtained by deleting an edge from a complete bipartite graph is determined by its spectrum. We provide some graphs, each of which is obtained from a complete bipartite graph by adding a vertex and an edge incident on the new vertex and an original vertex, which are not determined by their spectra.

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## 1. Introduction

The *adjacency matrix*  $A = (a_{ij})$  of a simple graph  $G$  is a 0-1 square matrix with rows and columns indexed by the vertex set  $V(G)$  of  $G$  such that for any  $i, j \in V(G)$ ,  $a_{ij} = 1$  iff  $i, j$  are adjacent in  $G$ . The *spectrum* of  $G$  is the set of eigenvalues of its adjacency matrix  $A$  together with their multiplicities. Two graphs are *cospectral* (also known as *isospectral*) if they share the same graph spectrum. To start our study, let us consider the smallest non-isomorphic cospectral graphs first given by Cvetković [7] as shown in Figure 1: the graph union  $K_{2,2} \cup K_1$  and the star graph  $K_{1,4}$ , where  $K_{p,q}$  denotes the complete bipartite graph of bipartition orders  $p$  and  $q$ . It is quick to check that their spectrum are both  $\{[0]^3, \pm 2\}$ . More constructions of cospectral graphs can be found in [16, 8, 18, 21, 10].

A graph  $G$  is *determined by the spectrum* if all the cospectral graphs of  $G$  are isomorphic to  $G$ . We abbreviate ‘determined by the spectrum’ to DS in the following. The question ‘which graphs are DS?’ goes back for more than half a century and originates from chemistry [17, 6], [9, Chapter 6]. After that, there appeared many examples and applications for the DS graphs. One of them is that in 1966 Fisher [15] modeled the shape of a drum by a



Figure 1: Two non-isomorphic cospectral graphs  $K_{2,2} \cup K_1$  and  $K_{1,4}$ .

graph from considering a question of Kac [19]: ‘Can you hear the shape of a drum?’ To see more details, Van Dam, Haemers and Brouwer gave a great amount of surveys for the DS graphs [10, 11, 12], [2, Chapter 14] in the past decades.

In [13] the non-regular bipartite graphs with four distinct eigenvalues were studied, and whether a such connected graph on at most 60 vertices is DS or not was determined. In [14] bipartite biregular graphs with 5 eigenvalues were studied, and all such connected graphs on at most 33 vertices were determined. In this research we study two families of nearly complete bipartite graphs one-edge different from a complete bipartite graph which also have 4 or 5 distinct eigenvalues without the assumptions of regularity, connectivity, or bounds on the number of vertices.

Let  $G$  be a simple bipartite graph with  $e$  edges. The *spectral radius*  $\rho(G)$  of  $G$  is the largest eigenvalue of the adjacency matrix of  $G$ . It was shown in [1, Proposition 2.1] that  $\rho(G) \leq \sqrt{e}$  with equality if and only if  $G$  is a complete bipartite graph with possibly some isolated vertices. It is straightforward to show that for any positive integer  $p$  the regular complete bipartite graph  $K_{p,p}$  is DS but, for example, the non-isomorphic bipartite graphs  $K_{1,6}$  and  $K_{2,3} \cup 2K_1$  are cospectral. There are several results extending [1, 4, 20, 5] of the above bound, which aim to solve an analog of the Brualdi-Hoffman conjecture for non-bipartite graphs [3], proposed in [1].

Our research is motivated from the following *twin primes bound* proposed in [5, Theorem 5.2]: For  $e \geq 4$ ,  $(e - 1, e + 1)$  is a pair of twin primes if and only if

$$\rho(e) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e - 1})}}{2}}$$

where  $\rho(e)$  denotes the maximal  $\rho(G)$  of a bipartite graph  $G$  on  $e$  edges which is not a union of a complete bipartite graph and possibly some isolated vertices. We need to introduce the notations  $K_{p,q}^-$  and  $K_{p,q}^+$  of the graphs which



Figure 2: The graphs  $K_{3,4}^-$  and  $K_{3,4}^+$  which are one-edge different from  $K_{3,4}$ .

are one-edge different from a complete bipartite graph. For  $2 \leq \min\{p, q\}$ , let  $K_{p,q}^-$  denote the graph with  $pq - 1$  edges obtained from  $K_{p,q}$  by deleting an edge, and  $K_{p,q}^+$  denote the graph with  $pq + 1$  edges obtained from  $K_{p,q}$  by adding a new vertex  $x$  and a new edge  $xy$  where  $y$  is a vertex in the partite set of order  $\min\{p, q\}$ . Note that  $K_{2,q}^+ = K_{2,q+1}^-$  for  $q \geq 2$ . Two examples of such graphs are shown in Figure 2.

The paper is organized as follows. Preliminary results are in Section 2. Theorem 10 in Section 3 proves that all the graphs  $K_{p,q}^-$  for  $2 \leq p \leq q$  are DS. Then Theorem 13 in Section 4 find all the pairs  $(p, q)$  such that the bipartite graph  $K_{p,q}^+$  is DS. Furthermore, for each  $K_{p,q}^+$  which is not DS we also find its unique non-isomorphic cospectral graph.

## 2. Preliminary

Basic results are provided in this section for later use.

**Lemma 1.** ([1, Proposition 2.1]) *Let  $G$  be a simple bipartite graph with  $e$  edges. Then*

$$\rho(G) \leq \sqrt{e}$$

*with equality iff  $G$  is a disjoint union of a complete bipartite graph and isolated vertices.*

The following result gives the relations between the spectrum and the numbers of vertices and edges in a graph which is proved simply by the definition of the adjacency matrix and its square.

**Proposition 2.** *Let  $G$  be a simple graph with eigenvalues  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ . Then*

- (i)  $G$  has  $n$  vertices, and

(ii)  $G$  has  $\frac{1}{2} \sum_{i=1}^n \lambda_i^2$  edges.

Since we focus on the bipartite graphs, a well-known spectral characterization of bipartite graphs [2, Proposition 3.4.1] is used in this paper.

**Proposition 3.** *A simple graph  $G$  is bipartite if and only if for each eigenvalue  $\lambda$  of  $G$ ,  $-\lambda$  is also an eigenvalue of  $G$  with the same multiplicity.*

Then a spectral characterization of complete bipartite graphs is straightforward.

**Proposition 4.** *Let  $G$  be a simple graph with spectrum  $\{[0]^{n-2}, \pm\lambda\}$  where  $n \geq 2$  is the number of vertices in  $G$ . Then  $\lambda^2$  is a nonnegative integer, and  $G$  is the union of some isolated vertices (if any) and a complete bipartite graph with  $\lambda^2$  edges.*

*Proof.* By Proposition 3 and Proposition 2 (ii),  $G$  is bipartite with  $\lambda^2$  edges. Since the equality holds in Lemma 1, the completeness follows.  $\square$

From Proposition 4 one can quickly find all the complete bipartite graphs which are DS.

**Corollary 5.** *For any positive integers  $p \leq q$ ,  $K_{p,q}$  is DS if and only if  $p' \leq p$  and  $q' \geq q$  for any positive integers  $p' \leq q'$  satisfying  $p'q' = pq$ .*

It is not difficult to compute the spectrum of each bipartite graph  $K_{p,q}^-$  or  $K_{p,q}^+$  [4, 20, 5].

**Proposition 6.** *Let  $2 \leq \min\{p, q\}$  be positive integers.*

(i) *The graph  $K_{p,q}^-$  has spectrum*

$$\left\{ [0]^{p+q-4}, \pm \sqrt{\frac{pq - 1 \pm \sqrt{(pq - 1)^2 - 4(p - 1)(q - 1)}}{2}} \right\}, \quad \text{and}$$

(ii) *the graph  $K_{p,q}^+$  has spectrum*

$$\left\{ [0]^{p+q-3}, \pm \sqrt{\frac{pq + 1 \pm \sqrt{(pq + 1)^2 - 4(p - 1)q}}{2}} \right\}.$$

We introduce some sets of bipartite graphs used in Lemma 8 and latter content.

**Definition 7.** Define the sets of bipartite graphs

$$\begin{aligned} \mathbb{K}^0 &:= \{K_{p,q} \mid p, q \in \mathbb{N}\}, \\ \mathbb{K}^- &:= \{K_{p,q}^- \mid 2 \leq p \leq q, (p, q) \neq (2, 2)\}, \\ \mathbb{K}^+ &:= \{K_{p,q}^+ \mid 2 \leq p \leq q\}, \text{ and} \\ \mathbb{K} &:= \mathbb{K}^0 \cup \mathbb{K}^- \cup \mathbb{K}^+. \end{aligned}$$

**Lemma 8.** ([5, Lemma 4.1]) *Let  $G$  be a simple bipartite graph on  $e$  edges without isolated vertices. If the spectral radius  $\rho(G)$  of  $G$  satisfies*

$$\rho(G) \geq \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e - 1})}}{2}},$$

then  $G \in \mathbb{K}$ .

The following result [11, Proposition 1] is well-known.

**Proposition 9.** *The path with  $n$  vertices is DS.*

### 3. Spectral characterizations of $K_{p,q}^-$

Note that the set  $\mathbb{K}^-$  of nearly bipartite graphs is defined in Definition 7. We prove that each graph  $G \in \{K_{2,2}^-\} \cup \mathbb{K}^-$  is DS in this section.

**Theorem 10.** *For any positive integers  $2 \leq p \leq q$ , the graph  $K_{p,q}^-$  is DS.*

*Proof.* If  $p = q = 2$  then  $K_{p,q}^-$  is a path on 4 vertices. Hence  $K_{2,2}^-$  is DS by Proposition 9. Let  $2 < q$  and  $G$  be a simple graph with the same spectrum as  $K_{p,q}^-$ . From Proposition 2, the numbers of vertices and edges in  $G$  are  $|V(G)| = p + q$  and  $|E(G)| = pq - 1$ . Additionally, Proposition 3 tells that  $G$  is a bipartite graph.

Suppose  $G$  has at least 2 nontrivial components  $G_1$  and  $G_2$ , where a nontrivial component is a connected graph with at least one edge. Then the spectra of  $G_1$  and  $G_2$  share the nonzero eigenvalues of  $G$ . Since  $G$  is bipartite,  $G_1$  and  $G_2$  are both bipartite. By Proposition 3 again and without loss of generality we have  $\text{sp}(G_1) = \{[0]^{m_1}, \pm e_1\}$  and  $\text{sp}(G_2) = \{[0]^{m_2}, \pm e_2\}$  for some nonnegative integers  $m_1, m_2$  with  $m_1 + m_2 + 4 \leq p + q$ , where

$$e_1 = \sqrt{\frac{pq - 1 + \sqrt{(pq - 1)^2 - 4(p - 1)(q - 1)}}{2}}$$

and

$$e_2 = \sqrt{\frac{pq - 1 - \sqrt{(pq - 1)^2 - 4(p - 1)(q - 1)}}{2}}$$

by Proposition 6 (i). From Proposition 4  $G_1$  is a complete bipartite graph with  $e_1^2$  edges, and thus  $(pq - 1)^2 - 4(p - 1)(q - 1)$  is a perfect square of type  $(pq - 1 - 2k)^2$  for some  $k \in \mathbb{N}$ . However,

$$\begin{aligned} (pq - 1)^2 - 4(p - 1)(q - 1) &= (pq - 1)^2 - 4(pq - p - q + 1) \\ &> (pq - 1)^2 - 4(pq - 2) \\ &= (pq - 3)^2, \end{aligned}$$

which is a contradiction.

Therefore  $G$  has exactly one nontrivial component  $G_0$ . Then

$$\text{sp}(G_0) = \left\{ [0]^m, \pm \sqrt{\frac{pq - 1 \pm \sqrt{(pq - 1)^2 - 4(p - 1)(q - 1)}}{2}} \right\}$$

for some nonnegative integer  $m$  with

$$(1) \quad |V(G_0)| = m + 4 \leq p + q.$$

Then by Proposition 2 (ii)

$$(2) \quad e := |E(G_0)| = |E(G)| = pq - 1.$$

Note that the spectral radius of  $G_0$

$$\begin{aligned} \rho(G_0) &= \sqrt{\frac{pq - 1 + \sqrt{(pq - 1)^2 - 4(p - 1)(q - 1)}}{2}} \\ &= \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - (p + q - 3))}}{2}} \\ &\geq \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e - 1})}}{2}}, \end{aligned}$$

since

$$(p + q - 3)^2 - (e - 1) = (p - 2)^2 + (q - 3)^2 + p(q - 2) - 2 \geq 0$$

for  $2 \leq p \leq q$  and  $3 \leq q$ . By Lemma 8  $G_0 \in \mathbb{K}$ . From Proposition 4  $G_0$  is not a complete bipartite graph. Hence  $G_0 \in \mathbb{K}^-$  or  $G_0 \in \mathbb{K}^+$ . Suppose  $G_0 \in \mathbb{K}^+$ , i.e.,  $G_0 = K_{p',q'}^+$  for some  $2 \leq p' \leq q'$ . Then by (1)

$$(3) \quad |V(G_0)| = p' + q' + 1 \leq p + q,$$

and by (2)

$$(4) \quad |E(G_0)| = p'q' + 1 = e = pq - 1.$$

According to Proposition 6 (ii),

$$(5) \quad (p' - 1)q' = (p - 1)(q - 1).$$

(4) and (5) imply  $q' + 3 = p + q$ . Then by (3)  $p' \leq 2$ , and hence  $p' = 2$ . Therefore  $G_0 = K_{2,q'}^+ = K_{2,q'+1}^-$  for some  $q' \geq 2$ , and we have  $G_0 \in \mathbb{K}^-$ . Let  $G_0 = K_{p'',q''}^-$  for some  $2 \leq p'' \leq q''$  and  $3 \leq q''$ . Then we rewrite the equations (3), (4) and (5) as

$$(6) \quad |V(G_0)| = p'' + q'' \leq p + q,$$

$$(7) \quad |E(G_0)| = p''q'' - 1 = pq - 1 \quad \text{and}$$

$$(8) \quad (p'' - 1)(q'' - 1) = (p - 1)(q - 1)$$

respectively, where the third equation (8) is from Proposition 6 (i). (7) and (8) imply  $|V(G_0)| = p'' + q'' = p + q = |V(G)|$ . Hence  $G_0 = G$ . The equalities in both sum and product of  $p'' \leq q''$  and  $p \leq q$  imply that  $(p'', q'') = (p, q)$ . Hence  $G = G_0 = K_{p,q}^-$ , and the result follows.  $\square$

**Remark 11.** From Theorem 10 we have  $K_{2,q}^+ = K_{2,q+1}^-$  is DS for  $2 \leq q$ . However, not all graphs in  $\mathbb{K}^+$  are DS. For example, the non-isomorphic graphs

$$K_{m+2,4m+2}^+ \quad \text{and} \quad K_{2m+2,2m+3}^- \cup mK_1$$

are cospectral for each  $m \in \mathbb{N}$ .

**Corollary 12.**  $K_{2,q}^+$  is DS for  $2 \leq q$ .  $\square$

#### 4. Spectral characterizations of $K_{p,q}^+$

Theorem 10 shows that for  $2 \leq p \leq q$  the graph  $K_{p,q}^-$  is DS and its proof appears useful for the study on the graph  $K_{p,q}^+$ . However, Remark 11 immediately gives a family of  $K_{p,q}^+$ 's which are not DS. In this section we present a

sufficient and necessary condition to determine whether  $K_{p,q}^+$  is DS or not for each pair  $(p, q)$ . Furthermore, we also find all the non-isomorphic cospectral graphs of every  $K_{p,q}^+$  which is not DS.

**Theorem 13.** *Let  $3 \leq p \leq q$  be positive integers. Then  $K_{p,q}^+$  is not DS if and only if the quadratic polynomial*

$$(9) \quad x^2 - (q + 3)x + (pq + 2) = 0$$

has two integral roots  $p'' \leq q''$  in  $[2, \infty)$ . Moreover, if  $K_{p,q}^+$  is not DS then  $K_{p'',q''}^- \cup (p - 2)K_1$  is its unique non-isomorphic cospectral graph.

*Proof.* Let  $G$  be a simple graph with the same spectrum as  $K_{p,q}^+$ . We prove Theorem 13 by two steps. In the first part of proof we show that  $G_0 \in \mathbb{K}^- \cup \mathbb{K}^+$  where  $G_0$  is obtained from  $G$  by deleting all the isolated vertices (if any). This process is similar to what we have done in the proof of Theorem 10. In the second part of proof, we prove that  $K_{p,q}^+$  is not DS if and only if (9) has two integral roots  $p'', q''$  and  $G_0 = K_{p'',q''}^-$  other than  $K_{p,q}^+$  itself. Moreover, if  $K_{p,q}^+$  is not DS then its only non-isomorphic cospectral graphs is obtained by adding  $p - 2$  many isolated vertices to  $K_{p'',q''}^-$ .

From Proposition 2, the numbers of vertices and edges in  $G$  are  $|V(G)| = p + q + 1$  and  $|E(G)| = pq + 1$ . Additionally, Proposition 3 tells that  $G$  is a bipartite graph. Suppose  $G$  has at least 2 nontrivial components  $G_1$  and  $G_2$ . Then the spectra of  $G_1$  and  $G_2$  share the nonzero eigenvalues of  $G$ . Since  $G$  is bipartite,  $G_1$  and  $G_2$  are both bipartite. By Proposition 3 and without loss of generality, we have  $\text{sp}(G_1) = \{[0]^{m_1}, \pm e_1\}$  and  $\text{sp}(G_2) = \{[0]^{m_2}, \pm e_2\}$  for some nonnegative integers  $m_1, m_2$  with  $m_1 + m_2 + 4 \leq p + q + 1$ , where

$$e_1 = \sqrt{\frac{pq + 1 + \sqrt{(pq + 1)^2 - 4(p - 1)q}}{2}}$$

and

$$e_2 = \sqrt{\frac{pq + 1 - \sqrt{(pq + 1)^2 - 4(p - 1)q}}{2}}$$

by Proposition 6 (ii). From Proposition 4,  $G_1$  is a complete bipartite graph with  $e_1^2$  edges, and thus  $(pq + 1)^2 - 4(p - 1)q$  is a perfect square of type  $(pq + 1 - 2k)^2$  for some  $k \in \mathbb{N}$ . However,

$$(pq + 1)^2 - 4(p - 1)q > (pq + 1)^2 - 4pq = (pq - 1)^2,$$

which is a contradiction.



Therefore  $G$  has exactly one nontrivial component  $G_0$ . Then

$$\text{sp}(G_0) = \left\{ [0]^m, \pm \sqrt{\frac{pq + 1 \pm \sqrt{(pq + 1)^2 - 4(p - 1)q}}{2}} \right\}$$

for some nonnegative integer  $m$  with

$$(10) \quad |V(G_0)| = m + 4 \leq p + q + 1.$$

Then by Proposition 2 (ii)

$$(11) \quad e := |E(G_0)| = |E(G)| = pq + 1.$$

Note that the spectral radius of  $G_0$

$$\begin{aligned} \rho(G_0) &= \sqrt{\frac{pq + 1 + \sqrt{(pq + 1)^2 - 4(p - 1)q}}{2}} \\ &= \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - q)}}{2}} \\ &\geq \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e - 1})}}{2}}, \end{aligned}$$

since

$$q^2 - (e - 1) = q^2 - pq = q(q - p) \geq 0$$

for  $3 \leq p \leq q$ . By Lemma 8  $G_0 \in \mathbb{K}$ . From Proposition 4,  $G_0$  is not a complete bipartite graph. Hence  $G_0 \in \mathbb{K}^-$  or  $G_0 \in \mathbb{K}^+$ . Here we complete the first part of proof.

Suppose  $G_0 \in \mathbb{K}^+$ , i.e.,  $G_0 = K_{p',q'}^+$  for some  $2 \leq p' \leq q'$ . Then by (10)

$$(12) \quad |V(G_0)| = p' + q' + 1 \leq p + q + 1,$$

and by (11)

$$(13) \quad |E(G_0)| = p'q' + 1 = e = pq + 1.$$

According to Proposition 6 (ii),

$$(14) \quad (p' - 1)q' = (p - 1)q.$$

(13) and (14) imply  $q' = q$  and  $p' = p$ . Therefore  $G = G_0 = K_{p,q}^+$ .

Suppose  $G_0 \in \mathbb{K}^-$ . Let  $G_0 = K_{p'',q''}^-$  for some  $2 \leq p'' \leq q''$  and  $3 \leq q''$ . Similar to the equations (12), (13) and (14) above,  $G$  is not DS if and only if there exists integral pair  $(p'', q'')$  that satisfies

$$(15) \quad |V(G_0)| = p'' + q'' \leq p + q + 1,$$

$$(16) \quad |E(G_0)| = p''q'' - 1 = pq + 1 \quad \text{and}$$

$$(17) \quad (p'' - 1)(q'' - 1) = (p - 1)q,$$

where the third equation (17) is from Proposition 6. Note that (16) and (17) imply

$$(18) \quad p'' + q'' = q + 3$$

and hence (15) automatically holds. Conversely, (16) and (18) imply (17). Hence the graph  $G_0 = K_{p'',q''}^-$  exists if and only if quadratic polynomial in (9) has two integral roots  $p'', q''$ . Note that  $G_0 = K_{p'',q''}^-$  is the only graph found in  $\mathbb{K}^- \cup \mathbb{K}^+$  except for  $K_{p,q}^+$ . Hence we conclude that for each pair of positive integers  $3 \leq p \leq q$ ,  $K_{p,q}^+$  is not DS if and only if (9) has two integral roots  $p'', q''$  and the only non-isomorphic cospectral graph is obtained from  $K_{p'',q''}^-$  by adding  $(p + q + 1) - (p'' + q'') = p - 2$  many isolated vertices by (18). Here we complete the second part of proof, and the result follows.  $\square$

The following lemma helps us to exhaustively enumerate  $K_{p,q}^+$  which has a non-isomorphic cospectral graph by using Theorem 13.

**Lemma 14.** *Let  $3 \leq p \leq q$  be integers. Then the quadratic polynomial in (9) has two integral roots  $p'', q'' \in [2, \infty)$  if and only if there exist nonnegative integers  $a, b, b', t$  with  $1 \leq b, b' < a$ ,  $\gcd(a, b) = 1$ ,  $bb' \equiv 1 \pmod{a}$ , and  $bb' + t \geq 2$  such that  $p, q, p'', q''$  can be written as*

$$(19) \quad p = b(a - b)t + bb' + \frac{b(1 - bb')}{a} + 1,$$

$$(20) \quad q = a^2t + ab',$$

$$(21) \quad p'' = bq/a + 1, \quad \text{and} \quad q'' = q + 2 - bq/a.$$

*Proof.* For the necessity, suppose that the quadratic polynomial in (9) has two integral roots  $p'', q'' \in [2, \infty)$ . Then  $p''q'' = pq + 2$  and  $p'' + q'' = q + 3$ . Thus

$$(22) \quad (p'' - 1) \cdot (q + 1 - (p'' - 1)) = (p - 1)q.$$

Note that  $p'' - 1 \leq q$ , otherwise we have  $p = 2$  which contradicts to  $3 \leq p$ . Let

$$(23) \quad p'' - 1 = \frac{bq}{a}$$

where  $1 \leq b < a$  are integers and  $\gcd(a, b) = 1$ . Then  $q \equiv 0 \pmod{a}$ . Let  $q = as$  for some  $s \in \mathbb{N}$ . Thus (22) becomes

$$(24) \quad \frac{b((a - b)s + 1)}{a} = p - 1.$$

Then  $(a - b)s + 1 \equiv 0 \pmod{a}$  since  $\gcd(a, b) = 1$ . Hence  $bs \equiv 1 \pmod{a}$ . Let  $s = at + b'$  where  $t$  and  $1 \leq b' < a$  are nonnegative integers with  $bb' = bs - bat \equiv 1 \pmod{a}$ . Therefore,  $q = as = a^2t + ab'$  as stated in (20). Substituting the above  $s = at + b'$  into (24), we have (19). The formulae of  $p''$  and  $q''$  are immediate from (23) and (18). Note that if  $t = 0$  and  $bb' = 1$  then  $p = 2$ , violating the assumption  $p \geq 3$ . Hence  $bb' + t \geq 2$ .

For the sufficiency, we check that for nonnegative integers  $a, b, b', t$  satisfying  $1 \leq b, b' < a$ ,  $\gcd(a, b) = 1$ ,  $bb' \equiv 1 \pmod{a}$ , and  $bb' + t \geq 2$ , the corresponding values of  $p, q, p'', q''$  are feasible. Note that we can rewrite (19) to (21) as

$$\begin{aligned} p &= \frac{b[(a - b)(at + b') + 1]}{a} + 1, \\ q &= a(at + b'), \\ p'' &= b(at + b') + 1, \quad \text{and} \quad q'' = (at + b')(a - b) + 2. \end{aligned}$$

One can immediately see that  $p'', q''$  are both integers not less than 2. Moreover, the sum and product of  $p'', q''$  are

$$\begin{cases} p'' + q'' = q + 3 \\ p''q'' = pq + 2 \end{cases},$$

which imply that  $p'', q''$  are the two integral roots of (9). □

To quickly find a non-isomorphic cospectral graphs pair which are nearly complete bipartite, a special case of Theorem 13 is provided in the following corollary.

**Corollary 15.** *For each pair of positive integers  $(t, a)$  with  $a \geq 2$ , the graph*

$$K_{(a-1)t+2, a^2t+a}^+$$

is not DS. Moreover,

$$K_{at+2,a(a-1)t+a+1}^- \cup (a-1)tK_1$$

is its unique cospectral graph.

*Proof.* Let  $b = b' = 1$  in Lemma 14. Then (19) to (21) become that

$$\begin{aligned} p &= (a-1)t + 2, \\ q &= a^2t + a, \\ p'' &= at + 2, \quad \text{and} \quad q'' = a(a-1)t + a + 1. \end{aligned}$$

Substituting these data into Theorem 13, we immediately have the proof.  $\square$

**Example 16.** By computer program, we list all  $K_{p,q}^+$ 's that are not DS for  $q \leq 20$  in the following table including the corresponding unique cospectral graphs and the values of parameters  $a, b, b', t$ . Note that the choices of  $(a, b, b', t)$  are not unique.

	The unique cospectral graph	$(a, b, b', t)$
$K_{3,6}^+$	$K_{4,5}^- \cup K_1$	$(2, 1, 1, 1)$ or $(3, 2, 2, 0)$
$K_{4,10}^+$	$K_{6,7}^- \cup 2K_1$	$(2, 1, 1, 2)$ or $(5, 3, 2, 0)$
$K_{5,14}^+$	$K_{8,9}^- \cup 3K_1$	$(2, 1, 1, 3)$ or $(7, 4, 2, 0)$
$K_{4,12}^+$	$K_{5,10}^- \cup 2K_1$	$(3, 1, 1, 1)$ or $(4, 3, 3, 0)$
$K_{5,15}^+$	$K_{7,11}^- \cup 3K_1$	$(3, 2, 2, 1)$ or $(5, 2, 3, 0)$
$K_{6,18}^+$	$K_{10,11}^- \cup 4K_1$	$(2, 1, 1, 4)$ or $(9, 5, 2, 0)$
$K_{5,20}^+$	$K_{6,17}^- \cup 3K_1$	$(4, 1, 1, 1)$ or $(5, 4, 4, 0)$

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