A characterization of bipartite distance-regular graphs*

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Abstract

It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular. Examples are given to show that the converse does not hold. Thus, a natural question is to find out when the converse is true. In this paper we give a quasi-spectral characterization of a connected bipartite 2-punctually distance-regular graph whose halved graphs are distance-regular. In the case the spectral diameter is even we show that the graph characterized above is distance-regular.

Keywords: Distance-regular graph, Distance matrices, Predistance polynomials, Spectral diameter, Spectral excess theorem

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1 Introduction

The study or characterizing the graphs whose eigenvalues and/or multiplicities satisfy a prescribed identity has a long history. For example, a well-known and real-world applicable theory asserts that a connected graph is bipartite if and only if its largest eigenvalue and smallest eigenvalue have the same absolute value. Recently, the eigenvectors, especially the one associated with the largest eigenvalue, are also taking into consideration, for instances, in mathematical theory: [15, 16, 12, 13, 10, 19]; in applications: [5, 2]. See [4, p. 65–69] for more applications. In this paper, we will give a (quasi-spectral) characterization of graphs when an identity involving eigenvalues, multiplicities, the eigenvector corresponding to the largest eigenvalue, and partial graph structure is satisfied. The details are as follows.

Throughout this paper, let G be a connected graph with vertex set V, order $n = |V|$, diameter D, and distance function ∂. The *adjacency matrix* A of G is the binary matrix indexed by V, where the entry $(A)_{uv} = 1$ if $\partial(u, v) = 1$, and $(A)_{uv} = 0$ otherwise. Assume that A has $d + 1$ distinct eigenvalues $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ with

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corresponding multiplicities $m_0 = 1$, m_1, ..., m_d. The *spectrum* of G is denoted by $\text{sp}\, G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, and the parameter d is called the *spectral diameter* of G. Note that $D \le d$ [1]. As is known, there is a sequence of orthogonal polynomials $p_0, p_1, \ldots, p_d$ with respect to the inner product $\langle\, , \rangle_\triangle$ (formally defined in the beginning of the next section), where $\deg p_i = i$ and $\langle p_i, p_i \rangle_\triangle = p_i(\lambda_0)$ for $0 \le i \le d$ [21]. Let α be the eigenvector of A associated with λ_0 such that $\alpha^t \alpha = n$ and all entries of α are positive. Note that α is usually called the *Perron vector*, and $\alpha = (1, 1, \ldots, 1)^t$ if and only if G is regular. For $u \in V$, let α_u be the entry corresponding to u in the eigenvector α. For $0 \le i \le d$, define the *weighted distance-i matrix* A_i of G to be the matrix indexed by V such that the entry $(A_i)_{uv} = \alpha_u \alpha_v$ if $\partial(u, v) = i$, and $(A_i)_{uv} = 0$ otherwise. In particular, for the case G is regular, A_i is binary and is the so-called *distance-i matrix* of G. For an integer $h \le d$, we say that G is *h-punctually distance-regular* if $A_h = p_h(A)$. Define $\delta_i = \sum_{u,v} (A_i \circ A_i)_{uv}/n$, where "$\circ$" is the entrywise product of matrices. A bipartite graph with bipartition (X, Y) is called *(k_1, k_2)-biregular* if every vertex in X has degree k_1 and every vertex in Y has degree k_2. The *distance-i graph* of G is the graph whose adjacency matrix is the distance-i matrix of G. For a connected bipartite graph G with bipartition (X, Y), the *halved graphs* G^X and G^Y are the two connected components of the distance-2 graph of G. It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular [3, Proposition 4.2.2]. Examples 5.1–5.3 show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Thus, a natural question is to find out when the converse is true. Our main result is the following.

Theorem 1.1. *Let G be a connected bipartite graph with bipartiton (X, Y) and spectral diameter d. Then the following conditions are equivalent.*

(i) $A_i = p_i(A)$ *for even i, where* $0 \le i \le d$;

(ii) $\delta_\ell = p_\ell(\lambda_0)$, *where $\ell = d - 1$ if d is odd, and $\ell = d$ otherwise;*

(iii) G is 2-punctually distance-regular and both halved graphs G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$.

Moreover, if (i)–(iii) hold and d is even, then G is distance-regular with diameter d.

In addition to the main result, we believe that Proposition 3.3, Theorem 3.4 and Proposition 4.5 are of independent interest.

This paper is organized as follows. In the next section we provide some simple but useful lemmas for bipartite graphs. In Section 3, we present some results related to the spectral excess theorem [12], and characterize the graphs with $\delta_i = p_i(\lambda_0)$ for $i \in \{0, 1\}$ (Propositions 3.7 and 3.9). In particular, these two propositions are very useful for checking the regularity or biregularity of a graph. In Section 4, we study the concepts of punctually distance-regularity and partially distance-regularity [9, 8]. In Section 5, we prove Theorem 1.1.

2 Some results for bipartite graphs

In this section we provide some simple but useful lemmas to be used later on. These results are related to the concept of orthogonal polynomials. The basic idea is to generalize the study of distance-regular graphs (see [1, 3, 22]).

2.1 Three-term recurrence

The *predistance polynomials* $p_0, p_1, \ldots, p_d$ of G are orthogonal polynomials with respect to the inner product

$$\langle p, q\rangle_\triangle := \sum_{i=0}^{d} \frac{m_i}{n} p(\lambda_i)q(\lambda_i) = \operatorname{tr}(p(A)q(A))/n,$$

where $\deg p_i = i$ and $\langle p_i, p_i\rangle_\triangle = p_i(\lambda_0)$ for $0 \le i \le d$ [21]. Moreover, they satisfy a three-term recurrence of the form

$$xp_i = c_{i+1}p_{i+1} + a_i p_i + b_{i-1}p_{i-1} \tag{1}$$

for $0 \le i \le d$, where c_{i+1}, a_i, b_{i-1} are scalars in $\mathbb{R}$, called the *preintersection numbers* of G, with $b_{-1} = c_{d+1} := 0$. If G is bipartite, then $a_i = 0$ for $0 \le i \le d$ [9], and thus $xp_i = c_{i+1}p_{i+1} + b_{i-1}p_{i-1}$. By this observation, the following lemma gives a three-term recurrence for bipartite graphs.

Lemma 2.1. *If G is bipartite, then the predistance polynomials satisfy a three-term recurrence of the form*

$$x^2 p_i = X_{i+2}p_{i+2} + Y_i p_i + Z_{i-2}p_{i-2} \tag{2}$$

for $0 \le i \le d$, where $X_{i+2} := c_{i+1}c_{i+2}$, $Y_i := b_i c_{i+1} + b_{i-1}c_i$ and $Z_{i-2} := b_{i-2}b_{i-1}$. □

Note that $a_i + b_i + c_i = \lambda_0$ for $0 \le i \le d$, where $c_0 := 0$ and $b_d := 0$ [6]. By directly computing, it follows that $X_i + Y_i + Z_i = \lambda_0^2$ for $0 \le i \le d$.

2.2 The "odd" or "even" part

The sum of all predistance polynomials gives the *Hoffman polynomial* H [17]:

$$H(x) := n \prod_{i=1}^{d} \frac{x - \lambda_i}{\lambda_0 - \lambda_i} = p_0 + p_1 + \cdots + p_d, \tag{3}$$

no matter whether the graph is regular or not. For a proof, see for instance [7]. Hoffman [17] proved that a connected graph G is regular if and only if $H(A) = J$, the all-ones matrix. The following lemma gives a generalization to nonregular graphs.

Lemma 2.2. (See [11, p. 117], [19, Lemma 2.1].) *Let G be a connected graph with adjacency matrix A and Perron vector α. Then, $H(A) = \alpha\alpha^t$. Moreover, G is regular if and only if $H(A) = J$, the all-ones matrix.* □

By Lemma 2.2 and (3), any connected graph G has the property that

$$A_0 + A_1 + \cdots + A_D = H(A) = p_0(A) + p_1(A) + \cdots + p_d(A). \tag{4}$$

If G is bipartite, then we can redescribe (4) (in Lemma 2.3) more precisely by only taking the "odd" or "even" part, which was also considered in [9]. Define $A^{\text{odd}} = \sum_{\text{odd } i} A_i$, $p^{\text{odd}} = \sum_{\text{odd } i} p_i$ and $\delta^{\text{odd}} = \sum_{\text{odd } i} \delta_i$. Similarly for A^{even}, p^{even} and δ^{even}. For two $n \times n$ real symmetric matrices M and N, define the inner product

$$\langle M, N \rangle := \frac{1}{n}\text{tr}(MN) = \frac{1}{n}\sum_{u,v}(M \circ N)_{uv}.$$

Then it follows that $\langle A^*, A^* \rangle = \delta^*$ and $\langle p^*(A), p^*(A) \rangle = p^*(\lambda_0)$ for $* \in \{\text{odd, even}\}$. If G is bipartite, then p_i is odd or even only depending on its degree i being odd or even [9]. The following lemma is proved by (4) and the fact that $(p_i(A))_{uv} = 0$ if $\partial(u, v)$ and i have distinct parity (since bipartite graphs contain no odd cycle).

Lemma 2.3. *If G is bipartite, then $A^{\text{odd}} = p^{\text{odd}}(A)$ and $A^{\text{even}} = p^{\text{even}}(A)$. Moreover, $\delta^{\text{odd}} = p^{\text{odd}}(\lambda_0)$ and $\delta^{\text{even}} = p^{\text{even}}(\lambda_0)$.* □

Remark 2.4. Observe that $p^{\text{odd}} = (H(x) - H(-x))/2$, $H(\lambda_0) = n$ and $H(\lambda_d) = 0$. Thus for bipartite graphs, we deduce that $\delta^{\text{odd}} = \delta^{\text{even}} = p^{\text{odd}}(\lambda_0) = p^{\text{even}}(\lambda_0) = n/2$.

3 The spectral excess theorem

The spectral excess theorem [12] asserts that $\delta_d \le p_d(\lambda_0)$ if G is regular, and equality is attained if and only if G is distance-regular. See [7, 14] for short proofs, and [9, 8] for some generalizations. The parameter $p_d(\lambda_0)$ is called the *spectral excess* of G, which can be expressed in terms of the spectrum, which is

$$p_d(\lambda_0) = \frac{n}{\pi_0^2}\left(\sum_{i=0}^{d}\frac{1}{m_i\pi_i^2}\right)^{-1},$$

where $\pi_i = \prod_{j\ne i}|\lambda_i - \lambda_j|$ for $0 \le i \le d$ [12]. The following lemma gives an expression of $p_{d-1}(\lambda_0)$ for bipartite graphs in terms of the spectrum. The proof is essentially identical to [7, p. 8–9], except the setting of the polynomials h_i.

Lemma 3.1. *Let G be a connected bipartite graph. Then*

$$p_{d-1}(\lambda_0) = n\left(2 + \sum_{i=1}^{d-1}\frac{(h_i(\lambda_0) + (-1)^{d-1}h_i(-\lambda_0))^2}{m_i h_i(\lambda_i)^2}\right)^{-1},$$

where $h_i = \prod_{j\ne 0,i,d}(x - \lambda_j)$ for $1 \le i \le d-1$. □

For $0 \le i \le d$, define $A_{\ge i} = \sum_{j\ge i} A_j$, $p_{\ge i} = \sum_{j\ge i} p_j$ and $\delta_{\ge i} = \sum_{j\ge i}\delta_j$. Similarly for $A_{\le i}$, $p_{\le i}$ and $\delta_{\le i}$. The parameter δ_D is referred to as the *average weighted excess* and $p_{\ge D}(\lambda_0)$ as the *generalized spectral excess* of G. Recently, the authors [19] proved the following "weighted" version of the spectral excess theorem for nonregular graphs. In fact, the approach of giving weights, the entries of the Perron vector, to the vertices of a nonregular graph has been recently used many times in the literature (see, for instance, [15, 16, 12, 13, 10]).

Theorem 3.2. (See [19].) *Let G be a connected graph with diameter D. Then $\delta_D \leq p_{\geq D}(\lambda_0)$ with equality if and only if $A_D = p_{\geq D}(A)$. Moreover, suppose further that $D = d$. Then equality holds if and only if G is distance-regular.* □

Define $A^{odd}_{\geq i} = \sum_{\text{odd } j \geq i} A_j$, $\delta^{odd}_{\geq i} = \sum_{\text{odd } j\geq i} \delta_j$ and $p^{odd}_{\geq i} = \sum_{\text{odd } j \geq i} p_j$. Similarly for A^*_Ω, δ^*_Ω and p^*_Ω, where $(\Omega, *) \in \{(\geq i, \text{even}), (\leq i, \text{odd}), (\leq i, \text{even})\}$. Summing the recurrence relation (1) from the terms with index $i+1$ to d, it follows that

$$xp_{\geq i+1} = b_i p_i + \lambda_0 p_{\geq i+1} - c_{i+1} p_{i+1}$$

[8, Proposition 2.5]. Note that if $A_{\geq i+1} = p_{\geq i+1}(A)$ and $\partial(u, v) < i$ for $u, v \in V$, then $(Ap_{\geq i+1}(A))_{uv} = 0 = (\lambda_0 p_{\geq i+1}(A))_{uv}$, and thus $b_i(p_i(A))_{uv} = c_{i+1}(p_{i+1}(A))_{uv}$. Using this fact, we have the following result.

Proposition 3.3. *Let G be a connected bipartite graph and $i \leq d-1$. Then $A_{\leq i} = p_{\leq i}(A)$ if and only if $A_j = p_j(A)$ for $0 \leq j \leq i$.*

Proof. The sufficiency is clear. To prove necessity, we only need to show that $A_i = p_i(A)$ (the remaining follows by similar argument). If $\partial(u, v) \geq i$, then $(A_i)_{uv} = (p_i(A))_{uv}$ by assumption. Suppose $\partial(u, v) < i$. If $\partial(u, v)$ and i have different parity, then $(A_i)_{uv} = 0 = (p_i(A))_{uv}$. Suppose that $\partial(u, v)$ and i have the same parity. Then $b_i(p_i(A))_{uv} = c_{i+1}(p_{i+1}(A))_{uv} = 0$. Since $b_i \neq 0$ for $i \leq d-1$, it follows that $(A_i)_{uv} = 0 = (p_i(A))_{uv}$. □

In [19], the authors posed the problem of characterizating the graphs which satisfy equality in Theorem 3.2 (or equivalently, $A_D = p_{\geq D}(A)$), and gave a simple solution: regular graphs with diameter 2 (in fact, these graphs are the so-called *distance-polynomial graphs* [23]). Under the condition $D = d$, such graphs are distance-regular (Theorem 3.2). Here we complete this characterization for bipartite graphs.

Theorem 3.4. *A connected bipartite graph with $A_D = p_{\geq D}(A)$ is distance-regular.*

Proof. Note that the assumption is equivalent to $A_{\leq D-1} = p_{\leq D-1}(A)$. By Proposition 3.3, $A_i = p_i(A)$ for $0 \leq i \leq D-1$. By Lemma 2.3, it follows that $p^*_{\geq D+1}(A)$ is the zero matrix, where $* \in \{\text{odd, even}\}$ has the same parity as $D+1$. This happens only for the case $D = d$, since otherwise $p^*_{\geq D+1}(\lambda_0) = 0$, contradicting the fact that $p_i(\lambda_0) > 0$ for $0 \leq i \leq d$. The remaining follows from Theorem 3.2. □

Let

$$\text{Proj}_N M := \frac{\langle N, M\rangle}{\langle N, N\rangle} N$$

denote the projection of M onto $\text{Span}\{N\}$. Lemmas 3.5–3.6 present some inequalities related to the spectral excess theorem. The proofs are essentially the same as in [14, Lemma 1].

Lemma 3.5. *Let G be a connected graph. For $0 \leq i \leq d$,*

(*i*) *$\delta_{\geq i} \leq p_{\geq i}(\lambda_0)$ with equality if and only if $A_{\geq i} = p_{\geq i}(A)$, and*

(*ii*) *$\delta_{\leq i} \geq p_{\leq i}(\lambda_0)$ with equality if and only if $A_{\leq i} = p_{\leq i}(A)$.* □

Lemma 3.6. *Let G be a connected bipartite graph. For $0 \le i \le d$ and $* \in$* {odd, even},

(i) $\delta^*_{\ge i} \le p^*_{\ge i}(\lambda_0)$ *with equality if and only if* $A^*_{\ge i} = p^*_{\ge i}(A)$, *and*

(ii) $\delta^*_{\le i} \ge p^*_{\le i}(\lambda_0)$ *with equality if and only if* $A^*_{\le i} = p^*_{\le i}(A)$. □

A natural question motivated by Lemmas 3.5–3.6 is to study the relation between the parameters δ_i and $p_i(\lambda_0)$ for $0 \le i \le d-1$ (the case $i = d$ is given in Theorem 3.2). We give some results in the following. Note that $p_0 = 1$. Proposition 3.7 is simple, but plays a crucial role in proving the regualrity of a graph, which follows from the inequality $\delta_{\le 0} \ge p_{\le 0}(\lambda_0)$ mentioned in Lemma 3.5. In fact, this result can also be derived by the Cauchy-Schwarz inequality: $\sum_{u\in V} \alpha_u^4 \ge (\sum_{u \in V} \alpha_u^2)^2/n = n$.

Proposition 3.7. *Let G be a connected graph. Then $\delta_0 \ge 1$ $(= p_0(\lambda_0))$ (which is equivalent to $\sum_{u\in V} \alpha_u^4 \ge n$), with equality if and only if any of the following conditions holds:*

(i) $A_0 = I$ $(= p_0(A))$,

(ii) *G is regular,*

(iii) *The Perron vector* $\alpha = (1, 1, \ldots, 1)^t$. □

Note that $p_1 = \lambda_0 x/\overline{k}$ (by the Gram-Schmidt procedure), where $\overline{k}$ is the average degree of G. Moreover, $A_1 = DAD$, where D is the diagonal matrix with entries $D_{uu} = \alpha_u$ for $u \in V$. For a connected graph G, we define its *weighted graph* G^w by giving the weight α_u to the vertex $u \in V$, and the weight $\alpha_u\alpha_v$ to the edge connecting u and v. Define in G^w the *degree* of u to be the sum of the weights $\alpha_u\alpha_v$ for those edges uv incident with u. Lemma 3.8 demonstrates that the average degree of the weighted graph equals the largest eigenvalue λ_0.

Lemma 3.8. $\langle A_1, A\rangle = \lambda_0$.

Proof. $\langle A_1, A\rangle = \dfrac{1}{n}\sum_{u,v}(A_1)_{uv} = \dfrac{1}{n}\,\mathbf{1}^t A_1 \mathbf{1} = \dfrac{1}{n}\,\mathbf{1}^t DAD\mathbf{1} = \lambda_0.$ □

The following proposition characterizes the graphs satisfying $\delta_1 = p_1(\lambda_0)$, which is useful for checking the regualrity or biregularity of a graph.

Proposition 3.9. *Let G be a connected graph. Then $\delta_1 \ge \lambda_0^2/\overline{k}$ $(= p_1(\lambda_0))$, with equality if and only if any of the following conditions holds:*

(i) $A_1 = p_1(A)$,

(ii) *G is regular or biregular.*

Proof. Computing $\mathrm{Proj}_{A_1} p_1(A)$ by the same argument as in [14, Lemma 1] and Lemma 3.8, it follows that $\delta_1 \ge p_1(\lambda_0)$, with equality if and only if $A_1 = p_1(A)$. Now it remains to show that $(i) \Leftrightarrow (ii)$. To prove necessity, we consider its weighted graph. Since $A_1 = p_1(A) = \lambda_0 A/\overline{k}$, all edges receive the same weight, $\lambda_0/\overline{k}$. If G is not bipartite, then it contains an odd cycle, and all vertices on this cycle must have the same weight. The assumption "G is connected" deduces that all vertices are of the same weight. Thus G is regular. For the case G is bipartite, the condition "all edges receive the same weight" implies that vertices in the same partite set have the same weight. Thus G is biregular. Now we prove sufficiency. If G is regular, then clearly $p_1(A) = \lambda_0 A/\overline{k} = A = A_1$. Suppose that G is (k_1, k_2)-biregular with bipartition (X, Y), where $|X| = n_1, |Y| = n_2$. Note that $\lambda_0 = \sqrt{k_1 k_2}$, $n_1 k_1 = n_2 k_2$ and the Perron vector

$$\alpha = (\underbrace{\alpha', \cdots, \alpha'}_{n_1} \underbrace{\alpha'', \cdots, \alpha''}_{n_2})^t,$$

where $\alpha' = \sqrt{\dfrac{n_1 + n_2}{2n_1}}$ and $\alpha'' = \sqrt{\dfrac{n_1 + n_2}{2n_2}}$. Thus

$$p_1(A) = \frac{\lambda_0}{\overline{k}} A = \frac{\sqrt{k_1 k_2}(n_1 + n_2)}{n_1 k_1 + n_2 k_2} A = \frac{n_1 + n_2}{2\sqrt{n_1 n_2}} A = \alpha' \alpha'' A = A_1. \qquad \square$$

The next question is to discuss the relation between δ_2 and $p_2(\lambda_0)$. We give the answer under the assumption G is regular, and provide an example to show that the regularity condition is necessary. Thus, there is no hope to determine the order of δ_2 and $p_2(\lambda_0)$ uniformly. Lemma 3.10 is proved by the inequality $\delta_{\le 2} \ge p_{\le 2}(\lambda_0)$ mentioned in Lemma 3.5, Proposition 3.7 and Proposition 3.9.

Lemma 3.10. *Let G be a connected regular graph. Then $\delta_2 \ge p_2(\lambda_0)$, with equality if and only if $A_2 = p_2(A)$.* $\square$

Example 3.11. (See [19].) Let P_3 be a path of three vertices, with spectrum $\{\sqrt{2}, 0, -\sqrt{2}\}$. Then $\delta_2 = 3/8 < 1/2 = p_2(\lambda_0)$.

4 Punctually and partially distance-regularity

The concepts of punctually distance-regularity and partially distance-regularity have been recently studied. In this paper, we study these two consepts, which are basically the same as in [9, 8], except that here we drop the regularity assumption, and the use of weighted distance matrices is taking into account. A connected graph is called *h-punctually distance-regualr* if $A_h = p_h(A)$; and is called *m-partially distance-regular* if $A_i = p_i(A)$ for $i \le m$. Note by Proposition 3.7 that the regularity condition is actually not necessary in the concept of partially distance-regularity. Clearly, the concepts of 0-punctually distance-regularity and 0-partially distance-regularity are identical. However, the 1-punctually distance-regularity and the 1-partially distance-regularity are not equivalent. For example, by Propositions 3.7 and 3.9, the path graph P_3 of three vertices is 1-punctually distance-regular, but not 1-partially distance-regular. The following lemma indicates that the concepts of 2-punctually distance-regularity and 2-partially distance-regularity coincide.

Proposition 4.1. *Let G be a connected graph. Then $A_2 = p_2(A)$ if and only if G is 2-partially distance-regular.*

Proof. We only need to prove necessity. Since $A_2 = p_2(A) = aA^2 + bA + cI$ for some real numbers a, b, c with $a \ne 0$, we conclude that A^2 has a constant diagonal, which implies that G is regular. The remaining follows from Propositions 3.7 and 3.9. $\square$

Proposition 4.2 states an equivalent condition of the 2-puctually distance-regularity for bipartite graphs with spectral diameter $d \geq 3$. Note that the assumption $d \geq 3$ is necessary, since otherwise the path graph P_3 of three vertices gives a counterexample. The proof follows from Proposition 3.3, Lemma 3.5 and Proposition 4.1.

Proposition 4.2. *Let G be a connected bipatrite graph with spectral diameter $d \geq 3$. Then $\delta_{\leq 2} = p_{\leq 2}(\lambda_0)$ if and only if G is 2-punctually distance-regular.* □

Lemma 4.3 demonstrates that for a connected bipartite 2-punctually distance-regular graph, its two halved graphs have the same spectrum (with appropriate spectral diameter), and, under further assumption, it gives a lower bound or exact value of the diameter, depending on the parity of its spectral diameter.

Lemma 4.3. *Let G be a connected bipartite graph with bipartition (X, Y), diameter D, spectral diameter d and $A_2 = p_2(A)$. Then the halved graphs G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Suppose further that at least one of G^X and G^Y has spectral diameter which is equal to its diameter. Then $D \geq d-1$ for odd d, and $D = d$ otherwise.*

Proof. Since G is bipartite, p_2 is even, that is, $p_2 = ax^2 + b$ for some real numbers a, b with $a \neq 0$. Let X_1 and Y_1 be adjacency matries of G^X and G^Y, respectively. Note that

$$A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$$

for some square matrix B (since G is regular by Proposition 4.1). Hence

$$\begin{pmatrix} X_1 & 0 \\ 0 & Y_1 \end{pmatrix} = A_2 = p_2(A) = aA^2 + bI = \begin{pmatrix} aBB^T + bI & 0 \\ 0 & aB^TB + bI \end{pmatrix}.$$

Since BB^T and B^TB have the same characteristic polynomial (see for instance [24, Theorem 2.8]), G^X and G^Y have the same spectrum. Note that if λ is an eigenvalue of A with eigenvector u then $a\lambda^2 + b$ is an eigenvalue of A_2 with the same eigenvector. Thus A_2 has $\lceil (d+1)/2 \rceil = \lfloor d/2 \rfloor + 1$ distinct eigenvalues, and so do G^X and G^Y. Hence G^X and G^Y are of spectral diameter $\lfloor d/2 \rfloor$. If at least one of G^X and G^Y has spectral diameter which is equal to its diameter, we derived that $d \geq D \geq 2 \cdot \lfloor d/2 \rfloor$, as claimed. □

In the end of this section, we give some results for connected bipartite graphs with $\delta_{d-1} = p_{d-1}(\lambda_0)$, which play crucial roles in the main result. Lemma 4.4 follows from Lemma 3.6. The proof of Proposition 4.5 is basically identical to [14, Proposition 2] (it is not difficult to prove this characterization by backward induction on i, using the recurrence relation (2), Lemma 2.3, Lemma 4.4, Proposition 3.7 and Proposition 3.9).

Lemma 4.4. *Let G be a connected bipartite graph. Then $\delta_{d-1} \leq p_{d-1}(\lambda_0)$, with equality if and only if $A_{d-1} = p_{d-1}(A)$.* □

Proposition 4.5. *Let G be a connected bipartite graph with $\delta_{d-1} = p_{d-1}(\lambda_0)$. Then $A_i = p_i(A)$ for all i with the opposite parity of d. In particular, G is regular if d is odd, and biregular otherwise.* □

5 Proof of the main result

It is well-known that the halved graphs of a bipartite distance-regular graph are distance-regular [3, Proposition 4.2.2]. We first provide three examples to show that the converse does not hold, that is, a connected bipartite graph whose halved graphs are distance-regular may not be distance-regular. Here we omit the computation details which are straightforward by definitions.

Example 5.1. (2-punctually distance-regular & odd spectral diameter)
Consider the Möbius-Kantor graph, i.e., the generalized Petersen graph $G(8,3)$ [20], with spectrum $\{3^1, \sqrt{3}^4, 1^3, (-1)^3, (-\sqrt{3})^4, (-3)^1\}$. Then $D = 4 < 5 = d$, $A_i = p_i(A)$ for $i \in \{0,1,2,4\}$, and both halved graphs are distance-regular with spectrum $\{6^1, 0^4, (-2)^3\}$.

Example 5.2. (not 2-punctually distance-regular & even spectral diameter)
Consider the Hoffman graph with spectrum $\{4^1, 2^4, 0^6, (-2)^4, (-4)^1\}$, which is cospectral to the Hamming 4-cube but not distance-regular [17, 3]. Then $D = d = 4$, $A_i = p_i(A)$ for $i \in \{0,1,3\}$, and its two halved graphs are the complete graph K_8 and the complete multipartite graph $K_{2,2,2,2}$, which are both distance-regular.

Example 5.3. (not 2-punctually distance-regular & odd spectral diameter)
Consider the graph obtained by deleting a 10-cycle from the complete bipartite graph $K_{5,5}$, with spectrum $\{3^1, ((\sqrt{5}+1)/2)^2, ((\sqrt{5}-1)/2)^2, ((-\sqrt{5}+1)/2)^2, ((-\sqrt{5}-1)/2)^2, (-3)^1\}$. Then $D = 3 < 5 = d$, $A_i = p_i(A)$ for $i \in \{0,1\}$, and both halves graphs are the complete graphs K_5, which are distance-regular.

Now we are ready to prove the main result, Theorem 1.1, which demonstrates that the converse is true under further assumptions: such a graph needs to be 2-punctually distance-regular with even spectral diameter.

Proof of Theorem 1.1. Conditions (i) and (ii) are equivalent by Proposition 4.5 (if d is odd) and Theorem 3.2 (if d is even). To complete the proof, we show that $(ii) \Leftrightarrow (iii)$. Recall that $\ell = d-1$ for odd d, and $\ell = d$ for even d. We first prove necessity. Since $\delta_\ell = p_\ell(\lambda_0)$, $A_j = p_j(A)$ for all even j by Proposition 4.5 (if d is odd) and Theorem 3.2 (if d is even). In particular, $A_0 = p_0(A) = I$ and $A_2 = p_2(A) = aA^2 + bI$ for some real numbers a, b with $a \neq 0$. Then G is regular and 2-punctually distance-regular. By Lemma 4.3, G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Since p_{2i} is even, we can assume $p_{2i} = f_i(ax^2 + b)$ for some $f_i \in \mathbb{R}[x]$ of degree i. Thus, for $0 \le i \le \lfloor d/2 \rfloor$,

$$\begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = A_{2i} = p_{2i}(A) = f_i(aA^2 + bI) = f_i(A_2) = \begin{pmatrix} f_i(X_1) & 0 \\ 0 & f_i(Y_1) \end{pmatrix},$$

where X_i and Y_i are distance-i matrices of G^X and G^Y, respectively. Therefore, G^X and G^Y are distance-regular with diameter $\lfloor d/2 \rfloor$. To prove sufficiency, first note by Proposition 4.1 that G is regular. By Lemma 4.3, G^X and G^Y have the same spectrum, and are of spectral diameter $\lfloor d/2 \rfloor$. Thus G^X and G^Y have the same (pre)distance-polynomials f'_i, $0 \le i \le \lfloor d/2 \rfloor$. Since G^X and G^Y are distance-regular,

$$A_{2i} = \begin{pmatrix} X_i & 0 \\ 0 & Y_i \end{pmatrix} = \begin{pmatrix} f'_i(X_1) & 0 \\ 0 & f'_i(Y_1) \end{pmatrix} = f'_i(A_2) = f'_i(p_2(A)) = g_{2i}(A)$$

for $0 \leq i \leq \lfloor d/2 \rfloor$, where X_i and Y_i are distance-i matrices of G^X and G^Y, respectively, and $g_{2i} \in \mathbb{R}[x]$ is even of degree $2i$. Since G is regualr, $A_\ell J = g_\ell(A)J = g_\ell(\lambda_0)J$. Then each row of A_ℓ has exactly $g_\ell(\lambda_0)$ ones, and thus $\delta_\ell = g_\ell(\lambda_0)$. Now it remains to show that $g_\ell = p_\ell$. Note that $\langle g_\ell, g_\ell \rangle_\triangle = \langle g_\ell(A), g_\ell(A) \rangle = \langle A_\ell, A_\ell \rangle = \delta_\ell = g_\ell(\lambda_0)$. For every polynomial $p \in \mathbb{R}_{\ell-1}[x]$, $\langle g_\ell, p \rangle_\triangle = \langle A_\ell, p(A) \rangle = 0$. By the uniqueness of the predistance polynomials, it follows that $g_\ell = p_\ell$. Moreover, if (i)–(iii) hold and d is even, then by Theorem 3.2, G is distance-regular with diameter d. □

Note that the Möbius-Kantor graph (Example 5.1) with odd spectral diameter satisfies Theorem 1.1 (i)–(iii) with $D = d - 1$. The following example shows that a bipartite graph with odd spectral diameter satisfying Theorem 1.1 (i)–(iii) and $D = d$ needs not to be distance-regular.

Example 5.4. Consider the regular bipartite graphs on 20 vertices obtained from the Desargues graph by the Godsil-McKay switching, which is not distance-regular with spectrum $\{3^1, 2^4, 1^5, (-1)^5, (-2)^4, (-3)^1\}$ [18]. Then $D = d = 5$, $A_i = p_i(A)$ for $i \in \{0, 1, 2, 4\}$, and both halved graphs are distance-regular with spectrum $\{6^1, 1^4, (-2)^5\}$.

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