

Combinatorial representations of Coxeter groups over a field of two elements ^{*}

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Abstract

Let W denote a simply-laced Coxeter group with n generators. We construct an n -dimensional representation ϕ of W over the finite field F_2 of two elements. The action of $\phi(W)$ on F_2^n by left multiplication is corresponding to a combinatorial structure extracted and generalized from Vogan diagrams. In each case W of types A, D and E, we determine the orbits of F_2^n under the action of $\phi(W)$, and find that the kernel of ϕ is the center $Z(W)$ of W .

1 Introduction

A *simply-laced Coxeter group* is a group $W_S(m)$ with a finite set of generators $S \subseteq W_S(m)$ subject only to relations

$$(ss')^{m(s,s')} = 1,$$

where $m(s, s) = 1$ and $m(s, s') = m(s', s) \in \{2, 3\}$ for $s \neq s'$ in S . When m is specified, we write W_S for $W_S(m)$, and if both S and m are specified, we write W for $W_S(m)$. A *Coxeter graph* S represents a simply-laced Coxeter group $W_S(m)$, and vice versa. The vertex set of this graph is S , and there is an edge joining two vertices s and s' whenever $m(s, s') = 3$. There are

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Coxeter groups which are not simple-laced. In this article we always assume simple-laced property in a Coxeter group to make the corresponding Coxeter graph S a simple graph.

We shall investigate a kind of *flipping puzzle*, which is also studied in [10, 11], associated with a given Coxeter graph S . The *configuration* of the flipping puzzle is S , together with an *assignment* of a unique state, white or black, on each vertex of S . A *move* in the puzzle is to select a vertex s which has black state, and then flip the state of each neighbor of s . When S is one of the Dynkin diagrams described in Figure 1, the configuration above is essentially a *Vogan diagram with identity involution*, which was first defined in [9], in a more general way, as a combinatorial object representing the real form of the corresponding complex simple Lie algebra and a system of choices. See also [1, 2, 5].

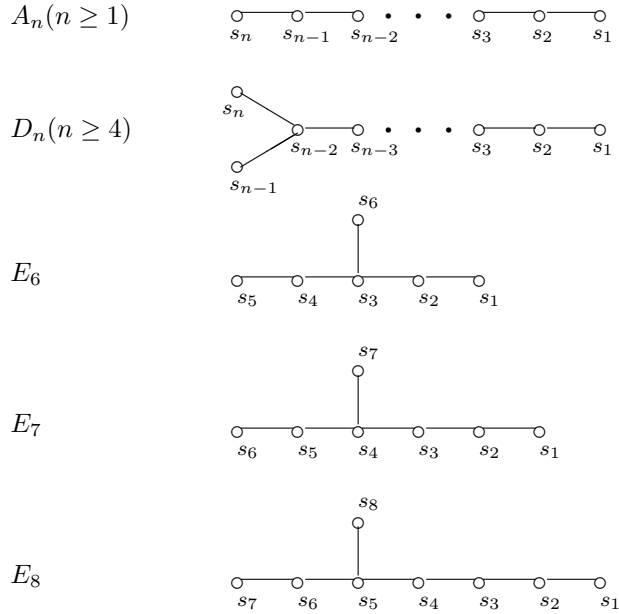


Figure 1: Simply-laced Dynkin diagrams.

We fix a simply-laced Coxeter group W and its Coxeter graph S , where $|S| = n$. Let F_2 denote the finite field of two elements 0 and 1. In Section 2, we use the column vector set $F_2^S = F_2^n$ to describe the set of configurations in the flipping puzzle associated with S by setting that $\ell_s = 1$ iff the configuration $\ell \in F_2^n$ has black state in the vertex s . For each vertex $s \in S$, we find a way to associate the move with selecting vertex s as an $n \times n$ invertible matrix \mathbf{s} over F_2 . This \mathbf{s} acts on a configuration $\ell \in F_2^n$ by left multiplication to become a new configuration $\mathbf{s}\ell$ which has the desired property as stated in the definition of the flipping puzzle when $\ell_s = 1$. Unlike in the definition, our move \mathbf{s} does not select configuration ℓ , but if a configuration has white state in s , it makes no effect; i.e. if $\ell_s = 0$ then $\mathbf{s}\ell = \ell$.

Let $\text{GL}_n(F_2)$ denote the set of $n \times n$ invertible matrices over F_2 and let \mathbf{W} denote the subgroup of $\text{GL}_n(F_2)$ generated by the moves \mathbf{s} for $s \in S$. We refer \mathbf{W} to a *flipping group* of S . In Section 3, we find that the canonical map $\phi : W \rightarrow \text{GL}_n(F_2)$, lifted from $\phi(s) = \mathbf{s}$ for $s \in S$, is a homomorphism with $\phi(W) = \mathbf{W}$. Due to its origination, we refer such a map to the *Vogan representation* of W . Then we find that the flipping group \mathbf{W} has trivial center in Section 4. In Sections 5, 6 and 7, we assume W to be A_n , D_n and E_n respectively. By using the finiteness of W , we can determine the size of the corresponding flipping group \mathbf{W} . We find that the kernel of the Vogan representation of W is the center $Z(W)$ of W when W is finite.

In the flipping puzzle on a Coxeter graph S , two configurations are said to be *equivalent* if one can be obtained from the other by a sequence of moves. Let \mathcal{P} denote the partition of configurations (i.e. F_2^n) according to the above equivalent relation. As a byproduct of our work, we solve the flipping puzzle associated with S when S is each of A_n , D_n and E_n by determining \mathcal{P} . Note that when S is a *tree*, a generalization of Dynkin diagrams, some partial results on \mathcal{P} are obtained in [11] and [10].

2 Flipping groups

Throughout this article, W will be a simply-laced Coxeter group with corresponding Coxeter graph S of n elements and edge set $R = \{ss' \mid m(s, s') = 3\}$. We shall construct a matrix group associated with the flipping puzzle on the Coxeter graph S . Let $\text{Mat}_n(F_2)$ denote the set of $n \times n$ matrices over F_2

with rows and columns indexed by S . Let F_2^n denote the set of n -dimensional column vectors over F_2 indexed by S . For $s \in S$, let \tilde{s} denote the characteristic vector of s in F_2^n ; that is $\tilde{s} = (0, 0, \dots, 0, 1, 0, \dots, 0)^t$, where 1 is in the position corresponding to s .

Definition 2.1. For $s \in S$, we associate a matrix $\mathbf{s} \in \text{Mat}_n(F_2)$, denoted by the bold type of s , as

$$\mathbf{s}_{uv} = \begin{cases} 1, & \text{if } u = v, \text{ or } v = s \text{ and } uv \in R; \\ 0, & \text{else,} \end{cases}$$

where $u, v \in S$.

The following is a reformulating of Definition 2.1.

Lemma 2.2. For $s, v \in S$,

$$\mathbf{s}\tilde{v} = \begin{cases} \tilde{v}, & \text{if } v \neq s; \\ \tilde{v} + \sum_{uv \in R} \tilde{u} & \text{if } v = s. \end{cases}$$

□

The flipping puzzle associated with S , which is described in the introduction, is now restated as follows. A configuration is simply a vector $\ell \in F_2^n$, where $\ell_s = 1$ (resp. $\ell_s = 0$) means that the vertex $s \in S$ has black state (resp. white state). In this setting, if $\ell_s = 1$ then $\mathbf{s}\ell$ is the new configuration after the move to select the vertex s . Note that if $\ell_s = 0$, we have $\mathbf{s}\ell = \ell$ from Lemma 2.2, so we can view the action of \mathbf{s} on ℓ as a *feigning move* on ℓ which is not originally defined as a move in the flipping puzzle. The following lemma is immediate from this combinatorial realization.

Lemma 2.3. For $s \in S$, \mathbf{s} is an involution; that is $\mathbf{s}^2 = I$, the identity matrix. □

From Lemma 2.3, \mathbf{s} is invertible, so we can give the following definition.

Definition 2.4. Let \mathbf{W} denote the subgroup of $\text{GL}_n(F_2)$ generated by the set $\{\mathbf{s} \mid s \in S\}$. \mathbf{W} is referring to the *flipping group* of S .

3 Coxeter groups and their combinatorial representations

Let W denote a simply-laced Coxeter group. Recall that an n -dimensional representation of W over F_2 is a homomorphism of W into $\mathrm{GL}_n(F_2)$. It is notorious difficult in the study of groups only defined by generators and relations. Hence the representation theory of Coxeter groups plays an important role in the study. In [7, Section 5.3], Humphreys gives "geometric representations" of Coxeter groups and use these representations to show that the finite Coxeter groups are essentially those associated with Dynkin diagrams. In this section we shall show that the flipping groups defined in the last section give "combinatorial representations" of simply-laced Coxeter groups. First we need a lemma.

Lemma 3.1. *Let W denote a simply-laced Coxeter group with Coxeter graph S . For $s \in S$, set $E_s \in \mathrm{Mat}_n(F_2)$ by*

$$E_s \tilde{v} = \begin{cases} 0, & \text{if } v \neq s; \\ \sum_{uv \in R} \tilde{u}, & \text{if } v = s \end{cases} \quad \text{for } v \in S. \quad (3.1)$$

Then with referring to the notation in Definition 2.1, the following (i)-(iii) hold.

- (i) $\mathbf{s} = I + E_s$ for $s \in S$
- (ii) $E_{s'} E_s = 0$, if $s's \notin R$.
- (iii) If $s_i s_{i-1} \in R$ for $i = 1, 2, \dots, t$, then

$$E_{s_t} E_{s_{t-1}} \cdots E_{s_0} = \begin{cases} E_{s_0}, & \text{if } s_t = s_0; \\ E_{s_t} E_{s_0}, & \text{if } s_t s_0 \in R. \end{cases}$$

Proof. (i) is immediate from Lemma 2.2. Note that $E_{s'} E_s \tilde{v} = 0$ by (3.1) for any $v, s, s' \in S$ with $s's \notin R$, and hence we have (ii). (iii) follows from the same reason as in (ii) by applying the product of matrices in either side of the equation to \tilde{v} and obtaining the desired equality in each case. \square

Theorem 3.2. *Let W denote a simply-laced Coxeter group with Coxeter graph S . Let \mathbf{W} denote the flipping group of S . Then there exists a surjective homomorphism $\phi : W \rightarrow \mathbf{W}$ such that $\phi(s) = \mathbf{s}$ for $s \in S$. In particular, ϕ is a representation of W over F_2 .*

Proof. We have seen $\mathbf{s}^2 = I$ for $s \in S$. It remains to show $(\mathbf{ss}')^2 = I$ if $s \neq s'$ and $ss' \notin R$, and to show $(\mathbf{ss}')^3 = I$ if $ss' \in R$. For $s, s' \in S$,

$$\begin{aligned}\mathbf{ss}' &= (I + E_s)(I + E_{s'}) \\ &= I + E_s + E_{s'} + E_s E_{s'}\end{aligned}$$

by Lemma 3.1(i). In the case $s \neq s'$ and $ss' \notin R$,

$$\begin{aligned}(\mathbf{ss}')^2 &= (I + E_s + E_{s'})(I + E_s + E_{s'}) \\ &= I + 2E_s + 2E_{s'} \\ &= I\end{aligned}$$

by Lemma 3.1(ii). In the case $ss' \in R$,

$$\begin{aligned}(\mathbf{ss}')^2 &= (I + E_s + E_{s'} + E_s E_{s'})(I + E_s + E_{s'} + E_s E_{s'}) \\ &= I + 3E_s + 3E_{s'} + 4E_s E_{s'} + E_{s'} E_s \\ &= I + E_s + E_{s'} + E_{s'} E_s\end{aligned}$$

and

$$\begin{aligned}(\mathbf{ss}')^3 &= (\mathbf{ss}')^2(\mathbf{ss}') \\ &= (I + E_s + E_{s'} + E_{s'} E_s)(I + E_s + E_{s'} + E_s E_{s'}) \\ &= I + 2E_s + 4E_{s'} + 2E_s E_{s'} + 2E_{s'} E_s \\ &= I\end{aligned}$$

by Lemma 3.1(iii). □

Definition 3.3. The representation ϕ defined in Theorem 3.2 is called the *Vogan representation* of W .

Suppose $J \subseteq S$. Let \mathbf{W}_J denote the subgroup of \mathbf{W} generated by the set $\{\mathbf{s} \mid s \in J\}$ and W_J denote simply-laced Coxeter group with the set J of generators with the function $m \upharpoonright J \times J$, the restriction of m to $J \times J$. Note that W_J is isomorphic to the subgroup of W generated by the set $\{s \mid s \in J\}$ [7, Section 5.5]. Hence we use the same symbol W_J to express these two isomorphic groups. It makes no confused if the place that W_J appears is also considered. For example, the first W_J in (iii) of the following lemma is in the first meaning and the remaining two W_J are in the second meaning. Note that \mathbf{W}_J , which is different to \mathbf{W}_J , is the flipping group on J . Let $G[J]$ denote the submatrix of $G \in \text{Mat}_n(F_2)$ with rows and columns indexed by J , and $\mathbf{W}_J[J] := \{G[J] \mid G \in \mathbf{W}_J\}$.

Lemma 3.4. *Suppose $J \subseteq S$. The following (i)-(iii) hold.*

(i) $\mathbf{W}_J[J] = \mathbf{W}_J$.

(ii) *The map $\psi : \mathbf{W}_J \rightarrow \mathbf{W}_J$, defined by $\psi(G) = G[J]$ for $G \in \mathbf{W}_J$, is a surjective homomorphism.*

(iii) *Let ϕ and ϕ' denote the Vogan representations of W_S and W_J respectively. Then $\phi' = \psi \circ \phi \upharpoonright W_J$. In particular, $\text{Ker } \phi \upharpoonright W_J \subseteq \text{Ker } \phi'$.*

Proof. By Definition 2.1, $\mathbf{s}_{uv} = 0$ for $s, u \in J$ and $v \in S - J$. By this, each matrix $G \in \mathbf{W}_J$ has the form

$$G = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$$

if indices in J are placed in the beginning of rows and columns, where A is a $|J| \times |J|$ matrix, B is an $(n - |J|) \times |J|$ matrix, C is an $(n - |J|) \times (n - |J|)$ matrix, and 0 is a $|J| \times (n - |J|)$ zero matrix. Then (i) and (ii) follow from the following matrix product rule in block form:

$$\begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \begin{pmatrix} A' & 0 \\ B' & C' \end{pmatrix} = \begin{pmatrix} AA' & 0 \\ BA' + CB' & CC' \end{pmatrix}.$$

Since $\psi \circ \phi(s) = \mathbf{s}[J] = \phi'(s)$ by (i) for all $s \in J$, we see $\phi' = \psi \circ \phi \upharpoonright W_J$, and this implies $\text{Ker } \phi \upharpoonright W_J \subseteq \text{Ker } \phi'$. \square

4 The center of a flipping group

As we shall see in Proposition 6.10 that the Coxeter group of type D_n has nontrivial center when n is even. In this section, we show that the center $Z(\mathbf{W})$ of any flipping group \mathbf{W} of a Coxeter graph S is trivial. Therefore, the center $Z(W)$ of any Coxeter group W is contained in the kernel of the Vogan representation of W . Recall that a Coxeter graph S is *disconnected* if there is a partition of $S = S' \cup S''$ with $S', S'' \neq \emptyset$ and there is no edge $uv \in R$ with $u \in S'$ and $v \in S''$. In this case the Coxeter group W is isomorphic to the direct product $W' \times W''$ of the Coxeter groups $W' = W_{S'}$ and $W'' = W_{S''}$. S is *connected* if S is not disconnected.

Proposition 4.1. *Let W denote a simple-laced Coxeter group with Coxeter graph S . Then the center $Z(\mathbf{W})$ of the flipping group \mathbf{W} of S is trivial.*

Proof. It suffices to assume that S is connected with at least two vertices. Let Z be an element in the center of \mathbf{W} and let u, v be two distinct elements in S . We show that $Z_{vu} = 0$ to conclude $Z = I$. Suppose $Z_{vu} = 1$. On the one hand $\mathbf{v}Z\tilde{u} \neq Z\tilde{u}$ since $Z\tilde{u}$ has 1 in the v th position. On the other hand, $\mathbf{v}Z\tilde{u} = Z\mathbf{v}\tilde{u} = Z\tilde{u}$ since $\mathbf{v}\tilde{u} = \tilde{u}$. Hence we have a contradiction. \square

From the above Proposition 4.1 we immediately have the following corollary.

Corollary 4.2. *Let W denote a simply-laced Coxeter group. Then the center $Z(W)$ is contained in the kernel of the Vogan representation of W .* \square

5 Coxeter groups of type A_n

Recall that the Vogan representation ϕ of W is *faithful* whenever ϕ is injective. Also ϕ is *irreducible* if there is no subspace $V \subseteq F_2^n$, $V \neq 0, F_2^n$, such that $\phi(W)V \subseteq V$. For $a \in F_2^n$, the subset of F_2^n consisting of all elements Ga with $G \in \phi(W)$ is called the *orbit* of F_2^n containing a under the action of $\phi(W)$.

In this section we assume that W is of type A_n with the Coxeter graph S as shown in Fig. 1, and determine the orbits of F_2^n under the action of $\phi(W)$. We also show that the kernel of the Vogan representation ϕ of W is the center $Z(W)$ of W and determine the reducibility of ϕ . The trivial case is given in the following.

Proposition 5.1. *Let W be a Coxeter group of type A_1 with the Vogan representation ϕ . Then the orbits of F_2 are $\{0\}$, $\{1\}$ under the action of $\phi(W)$, $\text{Ker } \phi = \{1, s_1\} = W = Z(W)$, and ϕ is irreducible.*

Proof. This follows from that $W = \{1, s_1\}$ and $\phi(W)$ is a trivial group. \square

In the remaining of this section, we always assume $n \geq 2$. Set

$$\bar{1} = \tilde{s}_1, \overline{i+1} = \mathbf{s}_i \mathbf{s}_{i-1} \cdots \mathbf{s}_1 \bar{1} \quad \text{for } 1 \leq i \leq n. \quad (5.1)$$

Note that

$$\bar{i} = \tilde{s}_{i-1} + \tilde{s}_i \quad \text{for } 2 \leq i \leq n, \quad (5.2)$$

and

$$\overline{n+1} = \tilde{s}_n = \bar{1} + \bar{2} + \cdots + \bar{n}. \quad (5.3)$$

Set $\Delta = \Delta(A_n) := \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. Note that Δ is a basis of F_2^n . We refer Δ to a *simple basis* of F_2^n . For $a \in F_2^n$, let $\Delta(a)$ denote the subset of Δ consisting of all the elements appeared in the expression of a as a linear combination of elements in Δ . The *weight* of an element $a \in F_2^n$ is $wt(a) := |\Delta(a)|$. For example, $\Delta(\overline{n+1}) = \Delta$ and $wt(\overline{n+1}) = n$.

Lemma 5.2. $\mathbf{s}_i \bar{i} = \overline{i+1}$, $\mathbf{s}_i \overline{i+1} = \bar{i}$ and \mathbf{s}_i fixes other vectors in $\{\bar{1}, \bar{2}, \dots, \overline{n+1}\} - \{\bar{i}, \overline{i+1}\}$ for $1 \leq i \leq n$.

Proof. This is immediate by applying Lemma 2.2, (5.1) and (5.2). \square

Let S_{n+1} denote the group of permutations on $\{\bar{1}, \bar{2}, \dots, \overline{n+1}\}$. By Lemma 5.2, we can give the following definition.

Definition 5.3. Let $\alpha : \mathbf{W} \rightarrow S_{n+1}$ be the homomorphism defined by

$$\alpha(G)\bar{j} = G\bar{j}$$

for each $1 \leq j \leq n+1$ and $G \in \mathbf{W}$.

Note that $\alpha(\mathbf{s}_i)$ is the transposition $(\bar{i}, \overline{i+1})$ in S_{n+1} for each $1 \leq i \leq n$.

Lemma 5.4. α is an isomorphism from \mathbf{W} onto S_{n+1} .

Proof. α is surjective since the transpositions $\alpha(\mathbf{s}_1), \alpha(\mathbf{s}_2), \dots, \alpha(\mathbf{s}_n)$ generate S_{n+1} . Since $\Delta \cup \{\overline{n+1}\}$ spans F_2^n , α is injective. \square

The next proposition determines the orbits of F_2^n under the action of \mathbf{W} .

Proposition 5.5. For $0 \leq i \leq \lfloor \frac{n+1}{2} \rfloor$,

$$O_i = \{a \in F_2^n \mid wt(a) = i \text{ or } n+1-i\}$$

is an orbit of F_2^n under the action of \mathbf{W} , where $\lfloor t \rfloor$ is the largest integer less than or equal to t .

Proof. Suppose $a \in F_2^n$ with $wt(a) = i$. Observe that from Lemma 5.4 and (5.3),

$$\Delta(Ga) = \begin{cases} \alpha(G)\Delta(a), & \text{if } \overline{n+1} \notin \alpha(G)\Delta(a); \\ \Delta - \alpha(G)\Delta(a), & \text{if } \overline{n+1} \in \alpha(G)\Delta(a) \end{cases}$$

for $G \in \mathbf{W}$. The proposition follows from this observation because the subgroup of $\alpha(\mathbf{W}) = S_{n+1}$ generated by the transpositions $\alpha(\mathbf{s}_1), \alpha(\mathbf{s}_2), \dots, \alpha(\mathbf{s}_{n-1})$ acts transitively on the fixed size subsets of Δ , and $\mathbf{s}_n \bar{n} = \bar{1} + \bar{2} + \dots + \bar{n}$ by Lemma 5.2 and (5.3). \square

In the following propositions, we study the reducibility of ϕ and $\text{Ker } \phi$.

Proposition 5.6. *The Vogan representation ϕ of W is irreducible if and only if n is even.*

Proof. Let V denote a nontrivial proper subspace of F_2^n such that $\phi(W)V \subseteq V$. Referring to Proposition 5.5, note that

$$V = \bigcup_{i \in J} O_i \quad (5.4)$$

for some proper subset $J \subseteq \{0, 1, \dots, \lfloor \frac{n+1}{2} \rfloor\}$ with $J \neq \{0\}$. Note that the set in the right side of (5.4) to be closed under addition is when it is the set of even weight vectors, and this occurs if and only if n is odd. \square

Proposition 5.7. *The Vogan representation ϕ of W is faithful. In particular, $\text{Ker } \phi = Z(W)$ is the trivial group.*

Proof. The first statement follows from that Proposition 5.4 and W is isomorphic to S_{n+1} [7, p41]. The second follows from the first and Corollary 4.2. \square

6 Coxeter groups of type D_n

Fix an integer $n \geq 4$. Let W denote the Coxeter group of type D_n with the Coxeter graph in Fig. 1. Let ϕ denote the Vogan representation of W , and $\mathbf{W} = \phi(W)$ be the flipping group of S . Set

$$\bar{1} = \tilde{s}_1, \quad \overline{i+1} = \mathbf{s}_i \mathbf{s}_{i-1} \cdots \mathbf{s}_1 \bar{1} \quad \text{for } 1 \leq i \leq n-1, \quad \text{and } \overline{n+1} = \tilde{s}_n. \quad (6.1)$$

Note that

$$\begin{aligned} \bar{i} &= \tilde{s}_{i-1} + \tilde{s}_i & \text{for } 2 \leq i \leq n-2, \\ \overline{n-1} &= \tilde{s}_{n-2} + \tilde{s}_{n-1} + \tilde{s}_n, \end{aligned} \quad (6.2)$$

and

$$\bar{n} = \tilde{s}_{n-1} + \tilde{s}_n = \bar{1} + \bar{2} + \cdots + \overline{n-1}. \quad (6.3)$$

Set $\Delta = \Delta(D_n) := \{\bar{1}, \bar{2}, \dots, \overline{n-1}, \bar{n}\}$ to be the simple basis of F_2^n in the case of type D_n . Set $\Delta(a)$ and $wt(a)$ as before for $a \in F_2^n$. For example, $\Delta(\bar{n}) = \Delta - \{\overline{n+1}\}$ by (6.3), and $wt(\bar{n}) = n-1$.

Lemma 6.1. *The following (i), (ii) hold.*

(i) *For each $1 \leq i \leq n-1$, $\mathbf{s}_i \bar{i} = \overline{i+1}$, $\mathbf{s}_i \overline{i+1} = \bar{i}$, and*

$$\mathbf{s}_i \bar{j} = \bar{j} \quad \text{for } j \in \{\bar{1}, \bar{2}, \dots, \overline{n+1}\} - \{\bar{i}, \overline{i+1}\}.$$

(ii) *$\mathbf{s}_n \overline{n-1} = \bar{n}$, $\mathbf{s}_n \bar{n} = \overline{n-1}$, $\mathbf{s}_n \overline{n+1} = \overline{n-1} + \bar{n} + \overline{n+1}$, and*

$$\mathbf{s}_n \bar{j} = \bar{j} \quad \text{for } j \in \{\bar{1}, \bar{2}, \dots, \overline{n-2}\}.$$

In particular, $\overline{n+1} \in \Delta(G\overline{n+1})$ and $G(\{\bar{1}, \bar{2}, \dots, \bar{n}\}) \subseteq \{\bar{1}, \bar{2}, \dots, \bar{n}\}$ for all $G \in \mathbf{W}$.

Proof. This follows immediately from Lemma 2.2, (6.1) and (6.2). \square

Let S_n denote the group of permutations on $\{\bar{1}, \bar{2}, \dots, \bar{n}\}$. By Lemma 6.1, we can give the following definition.

Definition 6.2. Let $\beta : \mathbf{W} \rightarrow S_n$ denote the homomorphism defined by

$$\beta(G)(\bar{j}) = G\bar{j}$$

for $1 \leq j \leq n$ and $G \in \mathbf{W}$.

In fact, β is surjective since the $n-1$ transpositions $\beta(\mathbf{s}_1), \beta(\mathbf{s}_2), \dots, \beta(\mathbf{s}_{n-1})$ generate S_n . Let Z denote the additive group of $(n-1)$ -dimensional subspace of F_2^n spanned by the set $\{\bar{1}, \bar{2}, \dots, \overline{n-1}\}$. Note that $a \in Z$ iff $\overline{n+1} \notin \Delta(a)$ for $a \in F_2^n$. By Lemma 6.1 and (6.3), Z is closed under the left multiplication of matrices in \mathbf{W} .

Proposition 6.3. *The Vogan representation ϕ of W is not irreducible. In particular $\phi(W)Z \subseteq Z$.* \square

Hence Z is a disjoint union of orbits of F_2^n under the action of \mathbf{W} . Note that $F_2^n - Z$ is also a disjoint union of orbits of F_2^n under the action of \mathbf{W} . The following proposition determines all the orbits of F_2^n under the action of \mathbf{W} .

Proposition 6.4. *The following are orbits of F_2^n under the action of \mathbf{W} .*

$$\begin{aligned} O_i &= \{a \in Z \mid wt(a) = i \text{ or } n-i\}, \\ \Omega_o &= \{a \in F_2^n - Z \mid wt(a) \equiv 1 \text{ or } n-1 \pmod{2}\}, \\ \Omega_e &= \{a \in F_2^n - Z \mid wt(a) \equiv 0 \text{ or } n \pmod{2}\}, \end{aligned}$$

where $0 \leq i \leq \lfloor \frac{n}{2} \rfloor$. In particular $\Omega_o = \Omega_e = F_2^n - Z$ is an orbit when n is odd.

Proof. The proof is similar to the proof of Proposition 5.5. The reason that O_i is an orbit follows from two facts: (i) $\beta(\mathbf{s}_1), \beta(\mathbf{s}_2), \dots, \beta(\mathbf{s}_{n-2})$ generate the subgroup S_{n-1} of S_n consisting of permutations on $\Delta - \{\overline{n+1}\}$ and S_{n-1} acts transitively on fixed size subsets of $\Delta - \{\overline{n+1}\}$, and (ii)

$$\mathbf{s}_{n-1}\overline{n-1} = \mathbf{s}_n\overline{n-1} = \overline{n} = \overline{1} + \overline{2} + \dots + \overline{n-1}$$

by Lemma 6.1(i),(ii) and (6.3). The reason that Ω_o and Ω_e are orbits follows from an additional fact that

$$wt(\mathbf{s}_n\overline{n+1}) = wt(\overline{n-1} + \overline{n} + \overline{n+1}) = wt(\overline{1} + \overline{2} + \dots + \overline{n-2} + \overline{n+1}) = n-1.$$

□

We study the structure of \mathbf{W} .

Definition 6.5. Let $\gamma : \mathbf{W} \rightarrow \text{Aut}(Z)$ denote the homomorphism from \mathbf{W} into the group $\text{Aut}(Z)$ of automorphisms of Z such that

$$\gamma(G)(u) = Gu$$

for $G \in \mathbf{W}$ and $u \in Z$.

Lemma 6.6. *There exists a homomorphism $\theta : S_n \rightarrow \text{Aut}(Z)$ such that $\gamma = \theta \circ \beta$.*

Proof. Since β is surjective, it suffices to show that the kernel of β is contained in the kernel of γ . Suppose $G \in \text{Ker } \beta$. Then $G\overline{i} = \overline{i}$ for $1 \leq i \leq n$; in particular G fixes each element in the basis $\Delta - \{\overline{n+1}\}$ of Z . Thus $G \in \text{Ker } \gamma$. □

Let $Z \rtimes_{\theta} S_n$ denote the group of *external semidirect product* of Z and S_n with respect to θ [8, p.155]; i.e. $Z \rtimes_{\theta} S_n$ is the set $Z \times S_n$ with the following product rule:

$$(u, \sigma)(v, \tau) = (u + \theta(\sigma)(v), \sigma\tau),$$

where $u, v \in Z$ and $\sigma, \tau \in S_n$. Note that $\overline{n+1} + G\overline{n+1} \in Z$ for any $G \in \mathbf{W}$ by Lemma 6.1.

Definition 6.7. Let $\delta : \mathbf{W} \rightarrow Z \rtimes_{\theta} S_n$ denote the map defined by

$$\delta(G) = (\overline{n+1} + G\overline{n+1}, \beta(G))$$

for any $G \in \mathbf{W}$.

Lemma 6.8. δ is an injective homomorphism of \mathbf{W} into $Z \rtimes_{\theta} S_n$.

Proof. For $G, H \in \mathbf{W}$,

$$\begin{aligned}
\delta(G)\delta(H) &= (\overline{n+1} + G\overline{n+1}, \beta(G))(\overline{n+1} + H\overline{n+1}, \beta(H)) \\
&= (\overline{n+1} + G\overline{n+1} + \theta(\beta(G))(\overline{n+1} + H\overline{n+1}), \beta(G)\beta(H)) \\
&= (\overline{n+1} + G\overline{n+1} + G(\overline{n+1} + H\overline{n+1}), \beta(G)\beta(H)) \\
&= (\overline{n+1} + GH\overline{n+1}, \beta(GH)) \\
&= \delta(GH).
\end{aligned}$$

This shows that δ is a homomorphism. δ is injective since if $\overline{n+1} + G\overline{n+1} = 0$ and $G \in \text{Ker } \beta$, then G fixes all vectors in Δ , so G is the identity matrix. \square

Note that $Z = \overline{n+1} + \Omega_o$ if n is odd, and $Z = (\overline{n+1} + \Omega_o) \cup (\overline{n+1} + \Omega_e)$ if n is even.

Lemma 6.9. $\delta(\mathbf{W}) = (\overline{n+1} + \Omega_o) \rtimes_{\theta} S_n$. In particular $\delta(\mathbf{W}) = Z \rtimes_{\theta} S_n$ if n is odd; $\delta(\mathbf{W})$ has index 2 in $Z \rtimes_{\theta} S_n$ if n is even.

Proof. Note that $\delta(\mathbf{s}_1), \delta(\mathbf{s}_2), \dots, \delta(\mathbf{s}_{n-2}), \delta(\mathbf{s}_{n-1})$ generate $0 \rtimes_{\theta} S_n$. Since Ω_o is an orbit containing $\overline{n+1}$, we have $\delta(\mathbf{W}) = (\overline{n+1} + \Omega_o) \rtimes_{\theta} S_n$. The second part follows from Proposition 6.4. \square

Proposition 6.10. The Vogan representation ϕ of W is faithful when n is odd; $\text{Ker } \phi$ has order 2 when n is even. Moreover, $\text{Ker } \phi$ is the center $Z(W)$ of W .

Proof. Note that W is isomorphic to the semidirect product $Z \rtimes S_n$ of Z and S_n [7, p.42]. By Lemma 6.9, ϕ is faithful when n is odd, and $\text{Ker } \phi$ has order 2 when n is even. From Corollary 4.2, $Z(W) \subseteq \text{Ker } \phi$, and from the fact that a normal subgroup of order 2 is contained in the center, we have $\text{Ker } \phi \subseteq Z(W)$. \square

7 Coxeter groups of type E_n

Fix an integer $n \geq 6$. Let W denote the Coxeter group of type E_n with the Coxeter graph S in Fig. 2. In this section we shall determine the orbits of F_2^n under the action of the flipping group \mathbf{W} of S . Restricting the attention

to the case $n = 6, 7$ or 8 in which W is finite, we show that the kernel of the Vogan representation ϕ of W is the center $Z(W)$ of W .

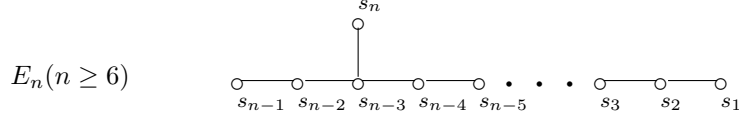


Figure 2: The Coxeter graph of type E_n .

Set $\bar{1} = \tilde{s}_1$, $\overline{i+1} = \mathbf{s}_i \mathbf{s}_{i-1} \cdots \mathbf{s}_1 \bar{1}$ for $1 \leq i \leq n-1$ and $\overline{n+1} = \tilde{s}_n$. Note that

$$\begin{aligned}
\bar{i} &= \tilde{s}_i + \tilde{s}_{i-1} \quad \text{for } 2 \leq i \leq n-3, \\
\overline{n-2} &= \tilde{s}_{n-3} + \tilde{s}_{n-2} + \tilde{s}_n, \\
\overline{n-1} &= \tilde{s}_{n-2} + \tilde{s}_{n-1} + \tilde{s}_n, \\
\bar{n} &= \tilde{s}_{n-1} + \tilde{s}_n.
\end{aligned} \tag{7.1}$$

Set $\Delta = \Delta(E_n) := \{\bar{1}, \bar{2}, \dots, \bar{n}\}$ to be the simple basis of F_2^n in this case. Observe that

$$\overline{n+1} = \bar{1} + \bar{2} + \cdots + \bar{n}. \tag{7.2}$$

Set $\Delta(a)$ and $wt(a)$ as before for $a \in F_2^n$. For example, $\Delta(\overline{n+1}) = \Delta$ and $wt(\overline{n+1}) = n$.

Lemma 7.1. *The following (i), (ii) hold.*

(i) For each $1 \leq i \leq n-1$, $\mathbf{s}_i \bar{i} = \overline{i+1}$, $\mathbf{s}_i \overline{i+1} = \bar{i}$, and

$$\mathbf{s}_i \bar{j} = \bar{j} \quad \text{for } \bar{j} \in \{\bar{1}, \bar{2}, \dots, \overline{n+1}\} - \{\bar{i}, \overline{i+1}\}.$$

(ii) $\mathbf{s}_n \overline{n+1} = \overline{n-2} + \overline{n-1} + \bar{n}$, $\mathbf{s}_n \bar{n} = \overline{n-2} + \overline{n-1} + \overline{n+1}$, $\mathbf{s}_n \overline{n-1} = \overline{n-2} + \bar{n} + \overline{n+1}$, $\mathbf{s}_n \overline{n-2} = \overline{n-1} + \bar{n} + \overline{n+1}$ and

$$\mathbf{s}_n \bar{j} = \bar{j} \quad \text{for } 1 \leq j \leq n-3.$$

Proof. This is immediate by applying Lemma 2.2 and (7.1). \square

Let S_n denote the group of permutations on $\Delta = \{\bar{1}, \bar{2}, \dots, \bar{n}\}$. Set $T := \{s_1, s_2, \dots, s_{n-1}\}$. Recall that \mathbf{W}_T is the subgroup of \mathbf{W} generated by $\{\mathbf{s} \mid s \in T\}$. By Lemma 7.1, we find that the set Δ is closed under the left multiplication of elements in \mathbf{W}_T .

Definition 7.2. Let $\epsilon : \mathbf{W}_T \rightarrow S_n$ denote the homomorphism satisfying

$$\epsilon(G)(\bar{j}) = G\bar{j}$$

for $1 \leq j \leq n$ and $G \in \mathbf{W}_T$.

In fact, ϵ is an isomorphism since Δ is a spanning set and the $n - 1$ transpositions $\epsilon(\mathbf{s}_1), \epsilon(\mathbf{s}_2), \dots, \epsilon(\mathbf{s}_{n-1})$ generate S_n .

Proposition 7.3. *The following are orbits of F_2^n under the action of \mathbf{W} .*

$$\begin{aligned} O_0 &= \{0\}, \\ O_1 &= \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 1 \text{ or } n - 2 \pmod{4}\}, \\ O_2 &= \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 2 \text{ or } n - 3 \pmod{4}\}, \\ O_3 &= \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 3 \text{ or } n \pmod{4}\}, \\ O_4 &= \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 0 \text{ or } n - 1 \pmod{4}\}. \end{aligned} \quad (7.3)$$

In particular $O_1 = O_3$ when $n \equiv 1 \pmod{4}$, $O_1 = O_4$ and $O_2 = O_3$ when $n \equiv 2 \pmod{4}$, $O_2 = O_4$ when $n \equiv 3 \pmod{4}$, and $O_1 = O_2$ and $O_3 = O_4$ when $n \equiv 0 \pmod{4}$.

Proof. It is clear that O_0 is an orbit. There are four cases to put nonzero vectors a, b in an orbit. (a) $wt(a) = wt(b)$: This is because $\epsilon(\mathbf{W}_T) = S_n$ acts transitively on the fixed size subsets of Δ ; (b) $wt(b) = n + 3 - wt(a)$, or $n - 1 - wt(a)$: This is from (a) and the observation that

$$wt(\mathbf{s}_n a) = \begin{cases} n + 3 - wt(a), & \text{if } |\Delta(a) \cap \{\bar{n}, \overline{n-1}, \overline{n-2}\}| = 3; \\ n - 1 - wt(a), & \text{if } |\Delta(a) \cap \{\bar{n}, \overline{n-1}, \overline{n-2}\}| = 1; \\ w(a), & \text{else} \end{cases} \quad (7.4)$$

by Lemma 7.1(ii) and (7.2); (c) $wt(a) = wt(b) - 4$: This is by applying the first case of (7.4) and then applying the second case of (7.4); and (d) $wt(a) = wt(b) + 4$: This is by applying the second case of (7.4) and then the first case of (7.4). The proposition follows from the above cases (a)-(d). \square

Similar to case of A_n , we determine the reducibility of ϕ from Proposition 7.3 immediately.

Proposition 7.4. *The Vogan representation ϕ is irreducible if and only if n is even. \square*

Recall that for $a \in F_2^n$, the *isotropy group* of a in \mathbf{W} is $\{G \in \mathbf{W} \mid Ga = a\}$, and the cardinality of the orbit of a is equal to the index of the isotropy group of a .

Corollary 7.5. *For $J := \{s_2, s_3, \dots, s_n\}$. the number $|\mathbf{W}_J||O_1|$ divides $|\mathbf{W}|$, where*

$$|O_1| = \begin{cases} 2^{n-1} - (-1)^{\frac{n}{4}} 2^{\frac{n-2}{2}}, & \text{if } n \equiv 0 \pmod{4}, \\ 2^{n-1}, & \text{if } n \equiv 1 \pmod{4}, \\ 2^{n-1} + (-1)^{\frac{n-2}{4}} 2^{\frac{n-2}{2}} - 1, & \text{if } n \equiv 2 \pmod{4}, \\ 2^{n-2} + (-1)^{\frac{n-3}{4}} 2^{\frac{n-3}{2}}, & \text{if } n \equiv 3 \pmod{4}. \end{cases} \quad (7.5)$$

Proof. Since \mathbf{W}_J is a subgroup of the isotropy group of $\bar{1}$, the number $|\mathbf{W}_J||O_1|$ divides $|\mathbf{W}|$. Note that by (7.3)

$$|O_1| = \begin{cases} \sum_{\substack{k \equiv 1, 2 \pmod{4} \\ 1 \leq k \leq n}} \binom{n}{k}, & \text{if } n \equiv 0 \pmod{4}, \\ \sum_{\substack{k \equiv 1 \pmod{2} \\ 1 \leq k \leq n}} \binom{n}{k}, & \text{if } n \equiv 1 \pmod{4}, \\ \sum_{\substack{k \equiv 0, 1 \pmod{4} \\ 1 \leq k \leq n}} \binom{n}{k}, & \text{if } n \equiv 2 \pmod{4}, \\ \sum_{\substack{k \equiv 1 \pmod{4} \\ 1 \leq k \leq n}} \binom{n}{k}, & \text{if } n \equiv 3 \pmod{4}, \end{cases}$$

where $\binom{n}{k}$ is the binomial coefficient. From this, we routinely prove (7.5) by induction on n . \square

We need to quote a lemma.

Lemma 7.6. (*[4, Lemma 10.2.11]*) *If W is of type E_7 or E_8 then $Z(W) = \{1, w_0\}$, where w_0 is the longest element of W . \square*

Recall that $T = \{s_1, s_2, \dots, s_{n-1}\}$ and $J = \{s_2, s_3, \dots, s_n\}$ when we indicate that the Coxeter group W is of type E_n .

Proposition 7.7. *The Vogan representation ϕ of W is faithful if W is of type E_6 , and $|\text{Ker } \phi| = 2$ if W is of type E_7 . Moreover, $\text{Ker } \phi = Z(W)$ if W is of type E_6 or E_7 .*

Proof. Suppose W is of type E_6 . With referring to Corollary 7.5, we have $|O_1| = 27$. By Lemma 3.4(iii) and Proposition 6.10 (the case D_5), we know $|\mathbf{W}_J| = 2^4 5!$, where J is of type D_5 . Since $|\mathbf{W}_J| |O_1|$ divides $|\mathbf{W}|$, we have $|\mathbf{W}| \geq 2^4 5! \cdot 27 = 2^7 3^4 5$. Since $|W| = 2^7 3^4 5$ [7, p.44], W is isomorphic to \mathbf{W} and $\text{Ker } \phi$ is trivial. By this and Corollary 4.2, $Z(W)$ is trivial.

Suppose W is of type E_7 . From Corollary 4.2 and Lemma 7.6, $|\text{Ker } \phi| \geq 2$. Since $|W| = 2^{10} 3^4 5 \cdot 7$ [7, p.44], we see that $|\mathbf{W}| \leq 2^9 3^4 5 \cdot 7$. On the other hand, according to a similar counting argument as above, we have $|O_1| = 28$, $|\mathbf{W}_J| = 2^7 3^4 5$, where J is of type E_6 , and hence $|\mathbf{W}| \geq 2^9 3^4 5 \cdot 7$. Thus, $|\mathbf{W}| = 2^9 3^4 5 \cdot 7$ and $|Z(W)| = |\text{Ker } \phi| = 2$. \square

We now go to the last case W of type E_8 . Note that $J = \{s_2, s_3, \dots, s_8\}$ is of type E_7 and $T \cap J = \{s_2, s_3, \dots, s_7\}$ is of type A_6 . We need more information of the nontrivial element w_0 in the center $Z(W_J)$ of W_J . It is quite complicate to describe w_0 directly as a product of elements in J . We borrow two notations to describe w_0 . Let ϕ denote the Vogan representation of W . Note that $\phi \upharpoonright W_{T \cap J}$ is an isomorphism of $W_{T \cap J}$ onto $\mathbf{W}_{T \cap J}$ by Lemma 3.4(ii) and Proposition 5.7. Also $\epsilon \upharpoonright \mathbf{W}_{T \cap J} : \mathbf{W}_{T \cap J} \rightarrow S_7$ is an isomorphism, where ϵ is as in Definition 7.2 and S_7 is the group of permutations on $\{\bar{2}, \bar{3}, \dots, \bar{8}\}$. The expression of w_0 is as follows.

$$\begin{aligned} w_0 = & \phi^{-1}(\epsilon^{-1}((\bar{2}, \bar{8}, \bar{3}, \bar{7}, \bar{4}, \bar{6}, \bar{5})))_{s_8} \phi^{-1}(\epsilon^{-1}((\bar{5}, \bar{8})(\bar{4}, \bar{7})(\bar{3}, \bar{6})))_{s_8} \\ & \phi^{-1}(\epsilon^{-1}((\bar{4}, \bar{8})(\bar{3}, \bar{7})(\bar{2}, \bar{6})))_{s_8} \phi^{-1}(\epsilon^{-1}((\bar{5}, \bar{8})(\bar{4}, \bar{7})))_{s_8} \\ & \phi^{-1}(\epsilon^{-1}((\bar{3}, \bar{7})(\bar{2}, \bar{6})))_{s_8}. \end{aligned} \quad (7.6)$$

It is routine to check that the above w_0 maps to $-I$ by the faithful representation defined in [3, Proposition 8] with $c = 0$ or in [6, p. 291] to conclude w_0 is in the center of W_J and indeed is the longest element of W_J by [3, Proposition 21]. Thus, we have the following lemma.

Lemma 7.8. *Let W be of type E_8 with the Vogan representation ϕ and $w_0 \in Z(W_J)$ be not identity. Then $\phi(w_0)$ is*

$$\begin{aligned} & \epsilon^{-1}((\bar{2}, \bar{8}, \bar{3}, \bar{7}, \bar{4}, \bar{6}, \bar{5}))_{s_8} \epsilon^{-1}((\bar{5}, \bar{8})(\bar{4}, \bar{7})(\bar{3}, \bar{6}))_{s_8} \epsilon^{-1}((\bar{4}, \bar{8})(\bar{3}, \bar{7})(\bar{2}, \bar{6}))_{s_8} \\ \times & \epsilon^{-1}((\bar{5}, \bar{8})(\bar{4}, \bar{7}))_{s_8} \epsilon^{-1}((\bar{3}, \bar{7})(\bar{2}, \bar{6}))_{s_8}. \end{aligned}$$

\square

Note that W_J is not isomorphic to its flipping group \mathbf{W}_J by Proposition 7.7. The following lemma claims that W_J is isomorphic to the subgroup \mathbf{W}_J of \mathbf{W} .

Lemma 7.9. *Let W be of type E_8 with the Vogan representation ϕ . Then the restriction $\phi \upharpoonright W_J$ of ϕ to J is injective.*

Proof. Let $\phi' : W_J \rightarrow \mathbf{W}_J$ denote the Vogan representation of W_J . From Lemma 3.4(iii) and Proposition 7.7, we see that $\text{Ker } \phi \upharpoonright W_J \subseteq \text{Ker } \phi' = \{1, w_0\}$, where w_0 is given in (7.6). To prove that $\text{Ker } \phi \upharpoonright W_J$ is trivial, it suffices to show that $\phi(w_0) \neq I$. This follows from the computation

$$\phi(w_0)\bar{8} = \bar{1} + \bar{8}$$

by applying the expression $\phi(w_0)$ in Lemma 7.8 to $\bar{8}$ and using Lemma 7.1 and (7.2) for $n = 8$ to simplify. \square

There is a similar result about W of type E_8 .

Proposition 7.10. *If W is of type E_8 then $\text{Ker } \phi$ has order 2. Moreover, $\text{Ker } \phi = Z(W)$.*

Proof. We have $|O_1| = 2^3 \cdot 3 \cdot 5$ from (7.5), $|\mathbf{W}_J| = |W_J| = 2^{10}3^45 \cdot 7$ from Lemma 7.9 and $|W| = 2^{14}3^55^27$ [7, p.44]. Therefore, as the proof of Proposition 7.7, $\text{Ker } \phi$ has order 2 and $\text{Ker } \phi = Z(W)$. \square

8 Concluding remarks

We list the main results of this article as follows.

Dynkin diagram	reducibility of ϕ	$ \text{Ker } \phi $
A_n	ϕ is irr. iff $n = 1$ or n is even.	$\begin{cases} 2, & \text{if } n = 1, \\ 1, & \text{else.} \end{cases}$
D_n ($n \geq 4$)	ϕ is not irr.	$\begin{cases} 2, & \text{if } n \text{ is even,} \\ 1, & \text{else.} \end{cases}$
E_6	ϕ is irr.	1
E_7	ϕ is not irr.	2
E_8	ϕ is irr.	2

Table 1: The reducibility and the kernel of a Vogan representation ϕ .

Coxeter graph	orbits
A_n	$O_i = \{a \in F_2^n \mid wt(a) = i \text{ or } n + 1 - i\} (0 \leq i \leq \lfloor \frac{n+1}{2} \rfloor)$.
D_n ($n \geq 4$)	$O_i = \{a \in Z \mid wt(a) = i \text{ or } n - i\} (0 \leq i \leq \lfloor \frac{n}{2} \rfloor)$, $\Omega_o = \{a \in F_2^n - Z \mid wt(a) \equiv 1 \text{ or } n - 1 \pmod{2}\}$, $\Omega_e = \{a \in F_2^n - Z \mid wt(a) \equiv 0 \text{ or } n \pmod{2}\}$, $\Omega_o = \Omega_e = F_2^n - Z$ when n is odd.
E_n ($n \geq 6$)	$O_0 = \{0\}$, $O_1 = \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 1 \text{ or } n - 2 \pmod{4}\}$, $O_2 = \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 2 \text{ or } n - 3 \pmod{4}\}$, $O_3 = \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 3 \text{ or } n \pmod{4}\}$, $O_4 = \{a \in F_2^n \mid a \neq 0, wt(a) \equiv 0 \text{ or } n - 1 \pmod{4}\}$. $O_1 = O_3$ when $n \equiv 1 \pmod{4}$, $O_1 = O_4$ and $O_2 = O_3$ when $n \equiv 2 \pmod{4}$, $O_2 = O_4$ when $n \equiv 3 \pmod{4}$, $O_1 = O_2$ and $O_3 = O_4$ when $n \equiv 0 \pmod{4}$.

Table 2: The orbits of F_2^n under the action of the flipping group of a Coxeter graph S .

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