D-bounded Distance-Regular Graphs

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Abstract

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$ and distance function δ. A (vertex) subgraph $\Delta \subseteq X$ is said to be **weak-geodetically closed** whenever for all $x, y \in \Delta$ and all $z \in X$,

$$\delta(x, z) + \delta(z, y) \leq \delta(x, y) + 1 \quad \longrightarrow \quad z \in \Delta.$$

Γ is said to be **D-bounded** whenever for all $x, y \in X$, x, y are contained in a common regular weak-geodetically closed subgraph of diameter $\delta(x, y)$. Assume Γ is D-bounded. Let $P(\Gamma)$ denote the poset whose elements are the weak-geodetically closed subgraphs of Γ with partial order by reverse inclusion. We obtain new inequalities for the intersection numbers of Γ; equality is obtained in each of these inequalities if and only if the intervals in $P(\Gamma)$ are modular. Moreover, we show this occurs if Γ has classical parameters and $D \geq 4$. We obtain the following corollary without assuming Γ to be D-bounded.

Corollary Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 4$. Suppose $b < -1$, and suppose the intersection numbers $a_1 \neq 0$, $c_2 > 1$. Then

$$\beta = \alpha \frac{1 + b^D}{1 - b}.$$

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1 Introduction

Let $\Gamma = (X, R)$ denote a distance-regular graph with distance function δ and diameter $D \geq 3$. A (vertex) subgraph $\Delta \subseteq X$ is said to be **weak-geodetically closed** whenever for all vertices $x, y \in \Delta$ and for all $z \in X$,

$$\delta(x,z) + \delta(z,y) \leq \delta(x,y) + 1 \quad \longrightarrow \quad z \in \Delta.$$

It turns out that if Δ is weak-geodetically closed and regular then Δ is distance-regular. For each integer $i \quad (0 \leq i \leq D)$, Γ is said to be i-**bounded** whenever for all $x, y \in X$ at distance $\delta(x,y) \leq i$, x, y are contained in a common regular weak-geodetically closed subgraph of Γ of diameter $\delta(x,y)$.

Now assume Γ is D-bounded. Let $P(\Gamma)$ denote the poset whose elements are the weak-geodetically closed subgraphs of Γ, with partial order induced by reverse inclusion. Using $P(\Gamma)$, we obtain the following inequalities for the intersection numbers of Γ :

$$\frac{b_{D-i-1} - b_{D-i+1}}{b_{D-i-1} - b_{D-i}} \geq \frac{b_{D-i-2} - b_{D-i}}{b_{D-i-2} - b_{D-i-1}} \qquad (1 \leq i \leq D-2). \tag{1}$$

We show equality is obtained in each of the above inequalities if and only if the intervals in $P(\Gamma)$ are modular. Moreover, we show this occurs if Γ has classical parameters and $D \geq 4$.

In [7], we assume $c_2 > 1$, $a_1 \neq 0$, and prove Γ to be D-bounded if some configurations do not exists (See Theorem 1.5 below). A special case of this is when Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with $D \geq 3$ and $b < -1$[7]. Furthermore, if we assume $D \geq 4$ then equalities in (1) hold. This leads to our main result, which we now state.

Theorem A Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 4$. Suppose $b < -1$, and suppose the intersection numbers $a_1 \neq 0$, $c_2 > 1$. Then

$$\beta = \alpha \frac{1 + b^D}{1 - b}.$$

We use Theorem A to obtain the following results, which we believe are of independent interest.

Theorem B Let Γ denote a distance-regular graph with diameter $D \geq 4$ and intersection number $c_2 > 1$. Then the following (i)-(ii) are equivalent.

(i) Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with $b = -a_1 - 1$.

(ii) Γ is the dual polar graph ${}^2A_{2D-1}(-b)$.

Theorem C Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 4$. Assume the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Suppose Γ is a near polygon graph. Then Γ is a dual polar graph or a Hamming graph.

Theorem D Let Γ denote a distance-regular graph with diameter $D \geq 4$, and the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Then the following (i)-(ii) are equivalent.

(i) Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with $b = -a_1 - 2$.

(ii) Γ is the Hermitian forms graph $Her_{-b}(D)$.

Using Hiroshi Suzuki's classification of D-bounded distance-regular graphs with $c_2 = 1$, $a_2 > a_1 > 1$[5], we prove the following result.

Theorem E There is no distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$, $D \geq 4$, $c_2 = 1$, and $a_2 > a_1 > 1$.

The paper is organized as follows.

Let Γ denote a D-bounded distance-regular graph of diameter $D \geq 3$. In section 2, we construct the poset $P(\Gamma)$, and prove (1)(See Proposition 2.9).

In section 3, we review the definition of distance-regular graphs with classical parameters $(D,\ b,\ \alpha,\ \beta)$, and study some basic properties of the constants b, α, β.

In section 4, we apply the results of section 2 to the case where Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 4$. In section 5, we prove Theorems A-D. In section 6, we prove Theorem E.

We now review some definitions, basic concepts, and results from our previous paper [7].

Let $\Gamma = (X, R)$ be a finite undirected, connected graph, without loops or multiple edges, with vertex set X, edge set R, path length distance function δ, and diameter $D := \max\{\delta(x, y) | x, y \in X\}$. Sometimes we write $\mathrm{diam}(\Gamma)$ to denote the diameter of Γ. By a **subgraph** of Γ, we mean a graph (Δ, Ξ), where Δ is a nonempty subset of X and $\Xi = \{ \{x, y\} | x, y \in \Delta, \{x, y\} \in R\}$. We refer to (Δ, Ξ) as the subgraph **induced on** Δ and by abuse of notation, we refer to this subgraph as Δ. For any $x \in X$ and any integer i, set

$$\Gamma_i(x) := \{y | y \in X, \delta(x, y) = i\}.$$

The **valency** $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is said to be **regular** with **valency** k whenever each vertex in X has valency k. For any $x \in X$, for any integer i, and for any $y \in \Gamma_i(x)$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y), \tag{2}$$
$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y), \tag{3}$$
$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y). \tag{4}$$

Γ is said to be **distance-regular** whenever for all integers i $(0 \le i \le D)$, and for all $x, y \in X$ with $\delta(x, y) = i$, the numbers

$$c_i := \Big|C(x, y)\Big|, \tag{5}$$
$$a_i := \Big|A(x, y)\Big|, \tag{6}$$
$$b_i := \Big|B(x, y)\Big| \tag{7}$$

are independent of x, y. The constants c_i, a_i, b_i $(0 \le i \le D)$ are known as the **intersection numbers** of Γ. Note that the valency $k = b_0$, $c_0 = 0$, $c_1 = 1$, $b_D = 0$, and

$$k = c_i + a_i + b_i \qquad (0 \le i \le D), \tag{8}$$

[1, p126].

From now on, let $\Gamma = (X, R)$ be a distance-regular graph with diameter D and intersection numbers c_i, a_i, b_i $(0 \le i \le D)$. We say a subgraph

$\Delta \subseteq X$ is **weak-geodetically closed** whenever for all vertices $x, y \in \Delta$, and for all $z \in X$,

$$\delta(x,z) + \delta(z,y) \leq \delta(x,y) + 1 \quad \longrightarrow \quad z \in \Delta.$$

Observe a subgraph $\Delta \subseteq X$ is weak-geodetically closed if and only if for all vertices $x, y \in \Delta$,

$$A(x,y) \cup C(x,y) \subseteq \Delta.$$

Definition 1.1 *Let Γ denote a distance-regular graph with diameter $D \geq 2$. Pick an integer $i \quad (0 \leq i \leq D)$. Γ is said to be i-**bounded**, if the following (i)-(ii) hold.*

(i) for all integers $j \quad (0 \leq j \leq i)$, and for all $x, y \in X$ such that $\delta(x,y) = j$, x, y are contained in a common weak-geodetically closed subgraph of diameter j.

(ii) All the weak-geodetically closed subgraphs of Γ with diameter $\leq i$ are regular.

Note 1.2 *Above definition is slightly different from [7]. Axiom (ii) above is not included in the definition of i-bounded in [7], since it holds automatically if $c_2 > 1$. Our present definition allows us to have a more general discussion.*

In Theorem 1.5, we characterize the i-bounded distance-regular graphs. First, we need a definition.

Definition 1.3 *Let Γ denote a graph with diameter $D \geq 2$. Pick an integer $i \quad (2 \leq i \leq D)$. By a **parallelogram** of length i in Γ, we mean a 4-tuple $xyzw$ of vertices of X such that*

$$\delta(x,y) = \delta(z,w) = 1, \quad \delta(x,w) = i,$$

$$\delta(x,z) = \delta(y,z) = \delta(y,w) = i - 1.$$

Lemma 1.4 *Let Γ denote a distance-regular graph with diameter $D \geq 2$. Pick an integer $i \quad (1 \leq i \leq D)$, and suppose Γ is i-bounded. Then Γ contains no parallelogram of length $\leq i + 1$.*

Proof. Suppose Γ contains a parallelogram $xyzw$ of some length $j \leq i+1$. Then $\delta(y,z) = j-1$. Let Δ denote a weak-geodetically closed subgraph containing y, z with diameter $j-1$. Observe that $x \in A(y,z)$ and $w \in A(z,y)$, so $x, w \in \Delta$. But $\delta(x,w) = j$, a contradiction.

With some restrictions on the intersection numbers, we get a stronger result.

Theorem 1.5 *([7, Theorem 6.4]) Let Γ denote a distance-regular graph with diameter $D \geq 2$, and assume the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Fix an integer $i \quad (1 \leq i \leq D)$. Then the following (i)-(ii) are equivalent.*

(i) Γ *is i-bounded.*

(ii) Γ *contains no parallelogram of length $\leq i+1$.*

For the rest of this section, we review some definitions about posets.

Let P denote a finite set. By a **partial order** on P, we mean a binary relation $\leq$ on P such that

(i) $p \leq p \qquad (\forall \ p \in P)$,

(ii) $p \leq q$ and $q \leq r \quad \longrightarrow \quad p \leq r \qquad (\forall \ p, q, r \in P)$,

(iii) $p \leq q$ and $q \leq p \quad \longrightarrow \quad p = q \qquad (\forall \ p, q \in P)$.

By a **partially ordered set** (or **poset**, for short), we mean a pair $(P, \leq)$, where P is a finite set, and where $\leq$ is a partial order on P. Abusing notation, we will suppress reference to $\leq$, and just write P instead of $(P, \leq)$.

Let P denote a poset, with partial order $\leq$, and let p and q denote any elements in P. As usual, we write $p < q$ whenever $p \leq q$ and $p \neq q$. We say q **covers** p whenever $p < q$, and there is no $r \in P$ such that $p < r < q$. An element $p \in P$ is said to be **maximal** (resp. **minimal**) whenever there is no $q \in P$ such that $p < q$ (resp. $q < p$). Let $\max(P)$ (resp. $\min(P)$) denote the set of all maximal (resp. minimal) elements in P. Whenever $\max(P)$ (resp. min (P)) consists of a single element, we denote it by 1 (resp. 0), and we say P has a 1 (resp. P has a 0).

Suppose P has a 0. By an **atom** in P, we mean an element in P that covers 0. We let A_P denote the set of atoms in P.

Suppose P has a 0. By a **rank function** on P, we mean a function

$$\mathrm{rank} : P \longrightarrow \mathbb{Z}$$

such that $\mathrm{rank}(0) = 0$, and such that for all $p, q \in P$,

$$q \ \text{ covers } \ p \qquad \longrightarrow \qquad \mathrm{rank}(q) - \mathrm{rank}(p) = 1. \tag{9}$$

Observe the rank function is unique if it exists. P is said to be **ranked** whenever P has a rank function. In this case, we set

$$\mathrm{rank}(P) := \max\{\mathrm{rank}(p) | p \in P\},$$

$$P_i := \{p | p \in P, \mathrm{rank}(p) = i\} \qquad (i \in \mathbb{Z}),$$

and observe $P_0 = \{0\}$, $P_1 = A_P$.

Let P denote any poset, and let S denote any subset of P. Then there is a unique partial order on S such that for all $p, q \in S$,

$$p \leq q \ \text{ (in } S) \qquad \longleftrightarrow \qquad p \leq q \ \text{ in } P.$$

This partial order is said to be **induced** from P. By a **subposet** of P, we mean a subset of P, together with the partial order induced from P. Pick any $p, q \in P$ such that $p \leq q$. By the **interval** $[p, q]$, we mean the subposet

$$[p, q] := \{r | r \in P, p \leq r \leq q\}$$

of P.

Let P denote any poset, and pick any $p, q \in P$. By a **lower bound** for p, q, we mean an element $r \in P$ such that $r \leq p$ and $r \leq q$. Suppose the subposet of lower bounds for p, q has a unique maximal element. In this case we denote this maximal element by $p \wedge q$, and say $p \wedge q$ **exists**. The element $p \wedge q$ is known as the **meet** of p and q. P is said to be (meet) **semi-lattice** whenever P is nonempty, and $p \wedge q$ exists for all $p, q \in P$. A semi-lattice has a 0. Suppose P is a semi-lattice, and pick $p, q \in P$. By a **upper bound** for p, q, we mean an element $r \in P$ such that $r \geq p$ and $r \geq q$. Observe the subposet of upper bounds for p, q is closed under $\wedge$; in particular, it has a unique minimal element if and only if it is nonempty. In this case we denote this minimal element by $p \vee q$, and say $p \vee q$ **exists**. The element $p \vee q$ is known as the **join** of p and q. By a lattice, we mean a semi-lattice P such that $p \vee q$ exists for all $p, q \in P$. A lattice has a 1.

Suppose P is a semi-lattice. Then every interval in P is a lattice.

Suppose P is a semi-lattice. Then P is said to be **atomic** whenever each element of P is a join of atoms. A semi-lattice P is atomic if and only if each element of P that is not 0 and not an atom covers at least 2 elements of P.

Suppose P is a lattice. Then P is said to be **upper semi-modular** (resp. **lower semi-modular**) whenever for all $p, q \in P$,

$$\begin{array}{rcl} p, q \text{ cover } p \wedge q & \longrightarrow & p \vee q \text{ covers } p, q \\ (\text{resp. } \; p \vee q \text{ covers } p, q & \longrightarrow & p, q \text{ cover } p \wedge q). \end{array}$$

P is said to be **modular** whenever P is upper semi-modular and lower semi-modular. Below are two examples of modular atomic lattices.

Example 1.6 *Let A denote a finite set. The* **Boolean algebra** B_A *is the poset of all subsets of A, ordered by inclusion. B_A is a modular atomic lattice of rank $|A|$. We often write $L_1(D) = B_A$, where $D = |A|$.*

Example 1.7 *Let V denote a finite vector space. The* **projective geometry** L_V *is the poset of all subspaces of V, ordered by inclusion. L_V is a modular atomic lattice of rank dim(V). We often write $L_q(D) = L_V$, where V is over the field $GF(q)$ and D is the dimension of V.*

2 The poset of a D-bounded distance-regular graph

Throughout this section, we fix a distance-regular graph $\Gamma = (X, R)$ with diameter $D \geq 3$. We assume Γ is D-bounded. We will construct a poset $P(\Gamma)$ which contains all the weak-geodetically closed subgraphs of Γ, and then study the properties of $P(\Gamma)$. We begin by quoting a lemma.

Lemma 2.1 *([7]) Let Γ denote a distance-regular graph with diameter D. Let c_j, a_j, b_j denote the intersection numbers of Γ. Then the following (i)-(ii) hold.*

(i) Let Δ denote a regular weak-geodetically closed subgraph. Then Δ is distance-regular, with intersection numbers

$$
\begin{aligned}
b_j(\Delta) &= b_j - b_d \quad (0 \leq j \leq d), &(10)\\
a_j(\Delta) &= a_j \quad (0 \leq j \leq d), &(11)\\
c_j(\Delta) &= c_j \quad (0 \leq j \leq d), &(12)
\end{aligned}
$$

where d denotes the diameter of Δ.

(ii) Let Δ, Δ' denote two regular weak-geodetically closed subgraphs of Γ and suppose $\Delta \subseteq \Delta'$. Then $\Delta = \Delta'$ if and only if Δ, Δ' have the same diameter.

Corollary 2.2 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded, and pick vertices $x, y \in X$. Then there exists a unique weak-geodetically closed subgraph containing x, y with diameter $\delta(x, y)$.*

Proof. The existence follows from Definition 1.1. Suppose Δ, Δ' are weak-geodetically closed subgraphs containing x, y with diameter $\delta(x, y)$. Observe $\Delta \cap \Delta'$ is also a weak-geodetically closed subgraph containing x, y with diameter $\delta(x, y)$. Hence by Lemma 2.1(ii),

$$
\begin{aligned}
\Delta &= \Delta \cap \Delta' \\
&= \Delta'.
\end{aligned}
$$

This proves the uniqueness.

Definition 2.3 *Let Γ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded. Let $P = P(\Gamma)$ denote the poset of all weak-geodetically closed subgraphs of Γ, with partial order $\leq$ satisfying*

$$\Delta \leq \Delta' \quad \longleftrightarrow \quad \Delta \supseteq \Delta' \quad (\forall \ \ \Delta, \Delta' \in P).$$

Lemma 2.4 *Let Γ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded, and set $P = P(\Gamma)$. Then the following (i)-(ii) hold.*

(i) P is ranked with rank D. Moreover,

$$\text{rank}(\Delta) = D - \text{diam}(\Delta) \quad (\forall \Delta \in P). \tag{13}$$

(ii) P is a (meet) semi-lattice. Moreover,

$$\Delta \wedge \Delta' = \bigcap \Omega, \tag{14}$$

where the intersection is over all weak geodetically closed subgraphs Ω that contain Δ, Δ'.

Proof. (i) We define rank as in (13) and observe $\text{rank}(X) = 0$. To prove (13) is a rank function, by (9), it remains to show for all $\Delta, \Delta' \in P$,

$$\Delta \ \text{ covers } \ \Delta' \qquad \longrightarrow \qquad \text{rank}(\Delta) - \text{rank}(\Delta') = 1. \tag{15}$$

Let Δ, Δ' be given and Δ covers Δ'. Note that

$$\Delta \subseteq \Delta', \quad \Delta' \neq \Delta, \tag{16}$$

so

$$\text{diam}(\Delta') - \text{diam}(\Delta) \geq 1 \tag{17}$$

by Lemma 2.1(ii). To prove (15), by (13), (17), it remains to show

$$\text{diam}(\Delta') - \text{diam}(\Delta) \leq 1. \tag{18}$$

Suppose

$$\text{diam}(\Delta') - \text{diam}(\Delta) > 1. \tag{19}$$

Set $\text{diam}(\Delta) = i$ and $\text{diam}(\Delta') = j$. Hence $j - i \geq 2$ by (19). We claim there exist vertices $x, y \in \Delta$ and $z \in \Delta' \setminus \Delta$ such that

$$\delta(x,y) = i, \quad \delta(y,z) = j - i, \quad \delta(x,z) = j. \tag{20}$$

Indeed, since $\text{diam}(\Delta) = i$, we can pick vertices $x, y \in \Delta$ with $\delta(x,y) = i$. Note that $x, y \in \Delta'$ by (16). Since Δ' is distance regular with diameter $j > i$, we can pick $z \in \Delta' \setminus \Delta$ satisfying (20). This proves our claim.

Pick $w \in C(z,y)$. Observe $w \in \Delta'$ and $w \in C(z,x)$ by construction. Hence by (19),

$$\begin{aligned}\delta(x,w) &= \delta(x,z)-1\\ &= j-1\\ &> i,\end{aligned}$$

so $w \in \Delta' \setminus \Delta$. Let Δ_1 be the weak-geodetically closed subgraph containing x, w with diameter $\delta(x,w)$. Observe that $\Delta \subseteq \Delta_1 \subseteq \Delta'$, $\Delta_1 \neq \Delta$ and $\Delta_1 \neq \Delta'$. This is impossible since Δ covers Δ'. This proves (i).

(ii) Fix Δ, $\Delta' \in P$. Note that $X \in P$ is a lower bound of Δ, Δ'. Observe that the intersection of weak-geodetically closed subgraphs is weak-geodetically closed. Hence $\Delta \wedge \Delta'$ exists; in fact, it is the intersection of all elements in P containing Δ, Δ'. This proves (ii).

Lemma 2.5 *Let Γ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded. Then each interval in $P = P(\Gamma)$ is lower semi-modular.*

Proof. Fix an interval $I \subseteq P$. To prove I is lower semi-modular, we fix $\Delta, \Delta' \in I$ such that $\Delta \vee \Delta'$ covers Δ, Δ', and show Δ, Δ' cover $\Delta \wedge \Delta'$. By Lemma 2.4(i) and the symmetry of Δ, Δ', it suffices to show

$$\operatorname{diam}(\Delta \wedge \Delta') - \operatorname{diam}(\Delta) = 1. \tag{21}$$

Note that

$$\Delta \vee \Delta' = \Delta \cap \Delta'. \tag{22}$$

Set

$$i := \operatorname{diam}(\Delta \vee \Delta'). \tag{23}$$

Note that

$$\operatorname{diam}(\Delta) = i+1 \tag{24}$$

and

$$\operatorname{diam}(\Delta') = i+1.$$

We claim there exist vertices $x, y \in \Delta \vee \Delta'$, $z \in \Delta \setminus \Delta'$, $w \in \Delta' \setminus \Delta$ such that

$$\delta(x,y) = i, \quad z \in B(y,x), \quad w \in B(x,y), \quad \delta(z,w) = i+2. \tag{25}$$

Indeed, by (23), we can pick vertices

$$x, y \in \Delta \vee \Delta' \tag{26}$$

with

$$\delta(x, y) = i. \tag{27}$$

Since $\mathrm{diam}(\Delta) = i + 1$, and since Δ is distance-regular by lemma 2.1(i), we can pick a vertex

$$z \in B(y, x) \cap \Delta. \tag{28}$$

Note that

$$\delta(x, z) = i + 1. \tag{29}$$

Observe that $z \notin \Delta'$ by (23), (26), (28), (29). Similarly, we can pick a vertex

$$w \in B(x, y) \cap \Delta' \tag{30}$$

and

$$w \notin \Delta. \tag{31}$$

Note that $w \in \Gamma_1(x) \setminus (A(x, z) \cup C(x, z))$ by (30), (28), (31) and construction. Hence $\delta(w, z) = i + 2$ by (29). This proves our claim.

Let Ω denote the weak-geodetically closed subgraph containing w, z such that

$$\mathrm{diam}(\Omega) = i + 2. \tag{32}$$

Observe that by (14)

$$\Omega \subseteq \Delta \wedge \Delta', \tag{33}$$

since $w, z \in \Delta \wedge \Delta'$. Also by construction,

$$\Delta \subseteq \Omega$$

and

$$\Delta' \subseteq \Omega.$$

Hence by (14) again,

$$\Delta \wedge \Delta' \subseteq \Omega. \tag{34}$$

By (33)-(34), we have

$$\Delta \wedge \Delta' = \Omega. \tag{35}$$

Now (21) follows from (35), (32), (24). This proves the lemma.

To study $P(\Gamma)$ further, we need the following fact about Γ.

Lemma 2.6 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded. Then*

$$b_i > b_{i+1} \quad (0 \leq i \leq D-1). \tag{36}$$

Proof. Fix vertices $x, y, z \in X$ such that $\delta(x, y) = i+1$, $\delta(z, y) = i$, and $\delta(x, z) = 1$. Let Δ (resp. Δ') denote the weak-geodetically closed subgraph containing x, y (resp. z, y) of diameter $i+1$ (resp. i). Observe $x \notin \Delta'$ and $\Delta' \subseteq \Delta$. Hence by (10)

$$\begin{aligned} b_i &= b_0 - b_0(\Delta') \\ &> b_0 - b_0(\Delta) \\ &= b_{i+1} \end{aligned}$$

as desired.

Lemma 2.7 *Let Γ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded, and set $P = P(\Gamma)$. Fix an integer $i \quad (1 \leq i \leq D-1)$. Then for all elements $\Delta \in P_{i-1}$, $\Delta' \in P_{i+1}$ with $\Delta \leq \Delta'$,*

$$\Big|[\Delta, \Delta'] \cap P_i\Big| = \frac{b_{D-i-1} - b_{D-i+1}}{b_{D-i-1} - b_{D-i}}. \tag{37}$$

We denote this number by f_i.

Proof. Note that $\mathrm{diam}(\Delta) = D-i+1$, $\mathrm{diam}(\Delta') = D-i-1$, and $\Delta' \subseteq \Delta$. Fix $x \in \Delta'$. We count the number $e := |\Gamma_1(x) \cap \Delta \setminus \Delta'\Big|$ in two ways to obtain (37). On the one hand, by (10),

$$\begin{aligned} e &= \Big|\Gamma_1(x) \cap \Delta \setminus (\Gamma_1(x) \cap \Delta')\Big| \\ &= (b_0 - b_{D-i+1}) - (b_0 - b_{D-i-1}) \\ &= b_{D-i-1} - b_{D-i+1}. \end{aligned} \tag{38}$$

On the other hand, observe that

$$\Delta_1 \cap \Delta_2 = \Delta' \quad (\forall \ \text{ distinct } \ \Delta_1, \Delta_2 \in [\Delta, \Delta'] \cap P_i).$$

Hence by (10),

$$\begin{aligned} e &= \sum_{\Delta_1 \in [\Delta,\Delta'] \cap P_i} \Big| \Gamma_1(x) \cap \Delta_1 \setminus (\Gamma_1(x) \cap \Delta') \Big| \\ &= \Big| [\Delta,\Delta'] \cap P_i \Big| ((b_0 - b_{D-i}) - (b_0 - b_{D-i-1})) \\ &= \Big| [\Delta,\Delta'] \cap P_i \Big| (b_{D-i-1} - b_{D-i}). \qquad (39) \end{aligned}$$

Now (37) follows from (38)-(39).

Corollary 2.8 *Let Γ denote a distance-regular graph with diameter D. Suppose Γ is D-bounded. Then the following (i)-(iii) hold.*

(i) $f_i > 1 \qquad (1 \leq i \leq D-1)$.

(ii) Each interval in $P(\Gamma)$ is atomic.

(iii) $P(\Gamma)$ is atomic.

Proof. (i) is clear from (36) and (37). (ii)-(iii) are clear from (i).

Proposition 2.9 *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose Γ is D-bounded, and set $P = P(\Gamma)$. Fix an integer $i \quad (1 \leq i \leq D-2)$. Then*

$$f_i \geq f_{i+1}. \qquad (40)$$

Furthermore, equality holds in (40) if and only if $[\Delta, \Delta']$ is modular for all $\Delta \in P_{i-1}$, $\Delta' \in P_{i+2}$ with $\Delta \leq \Delta'$.

Proof. Pick $\Delta \in P_{i-1}$, $\Delta' \in P_{i+2}$ such that $\Delta \leq \Delta'$, and set

$$\begin{aligned} A &:= [\Delta,\Delta'] \cap P_i, \\ B &:= [\Delta,\Delta'] \cap P_{i+1}. \end{aligned}$$

By Lemma 2.7 and a routine counting argument,

$$\Big|A\Big| f_{i+1} = \Big|B\Big| f_i. \qquad (41)$$

First, we count $\Big|B\Big|$. To do this, pick distinct Δ_1, $\Delta_2 \in B$. Observe

$$\Delta_1 \vee \Delta_2 = \Delta',$$

so by Lemma 2.5,

$$\Delta_1 \wedge \Delta_2 \in A.$$

This proves

$$\begin{aligned} \Big|B\Big| &= \Big|\{\Delta_1\}\Big| + \Big|\{\Delta_2 \in B | \Delta_1 \wedge \Delta_2 \in A\}\Big| \\ &= 1 + f_i(f_{i+1} - 1). \end{aligned} \tag{42}$$

Next, pick $\Delta_3 \in A$, and observe that

$$\begin{aligned} \Big|A\Big| &\geq \Big|\{\Delta_3\}\Big| + \Big|\{\Delta_4 \in A | \Delta_4 \vee \Delta_3 \in B\}\Big| \\ &= 1 + f_{i+1}(f_i - 1), \end{aligned} \tag{43}$$

with equality holds iff $[\Delta, \Delta']$ is upper semi-modular.

Combining (41)-(43) and simplifying, we find

$$(f_i - f_{i+1})(f_{i+1} - 1)(f_i - 1) \geq 0, \tag{44}$$

with equality hold if and only if $[\Delta, \Delta']$ is upper semi-modular. It follows

$$f_i \geq f_{i+1}, \tag{45}$$

since $f_{i+1} > 1$, $f_i > 1$. Note that equality in (45) holds if and only if $[\Delta, \Delta']$ is modular by Lemma 2.5. This proves the Lemma.

Proposition 2.10 *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Assume Γ is D-bounded, and set $P = P(\Gamma)$. Then the following (i)-(iii) are equivalent.*

(i) *Equality holds in (40), for all integers i $(1 \leq i \leq D-2)$.*

(ii) *Every rank 3 interval in P is modular.*

(iii) *Every interval in P is modular.*

Proof. (i)$\longrightarrow$(ii). This is immediate from Proposition 2.9.

(ii)$\longrightarrow$(iii). By Lemma 2.5, it remains to prove each interval is upper semi-modular. Suppose this is not true. We pick an interval I which is not upper semi-modular with minimal rank r. Note that $r \geq 4$. Pick Δ, $\Delta' \in I$ such that Δ, Δ' cover $\Delta \wedge \Delta'$, but $\Delta \vee \Delta'$ does not cover Δ, Δ'. Note that $I = [\Delta\wedge\Delta', \Delta\vee\Delta']$, otherwise $\mathrm{rank}([\Delta\wedge\Delta', \Delta\vee\Delta']) < r$ and $\Delta\vee\Delta'$ covers Δ, Δ' by construction, a contradiction. Pick $\Delta_1 \in [\Delta, \Delta \vee \Delta']$ such that $\Delta \vee \Delta'$ covers Δ_1. Pick $\Delta_2 \in [\Delta', \Delta \vee \Delta']$ such that $\Delta \vee \Delta'$ covers Δ_2. Observe that $\Delta_1 \vee \Delta_2 = \Delta \vee \Delta'$ covers Δ_1, Δ_2. Then by Lemma 2.5, Δ_1, Δ_2 cover $\Delta_1 \wedge \Delta_2$. Note that $\mathrm{rank}([\Delta \wedge \Delta', \Delta_1 \wedge \Delta_2]) \geq 2$. Pick $\Delta_3 \in [\Delta \wedge \Delta', \Delta_1 \wedge \Delta_2]$ such that Δ_3 covers $\Delta \wedge \Delta'$. Observe that $\Delta, \Delta_3 \in [\Delta \wedge \Delta', \Delta_1]$,

$$\Delta \wedge \Delta_3 = \Delta \wedge \Delta',$$

and $\mathrm{rank}([\Delta \wedge \Delta', \Delta_1]) < r$. Hence by construction, $\Delta \vee \Delta_3$ covers Δ_3. Similarly, $\Delta' \vee \Delta_3$ covers Δ_3. Note that $\mathrm{rank}([\Delta_3, \Delta \vee \Delta']) < r$, and

$$(\Delta \vee \Delta_3) \wedge (\Delta' \vee \Delta_3) = \Delta_3.$$

Hence by the construction, $(\Delta \vee \Delta_3) \vee (\Delta' \vee \Delta_3)$ covers $\Delta \vee \Delta_3$, $\Delta' \vee \Delta_3$. Observe that

$$\begin{aligned}\mathrm{rank}([\Delta \wedge \Delta', \Delta \vee \Delta']) &\leq \mathrm{rank}([\Delta \wedge \Delta', \Delta \vee \Delta' \vee \Delta_3]) \\ &= 3,\end{aligned}$$

a contradiction.

(iii)$\longrightarrow$(i). This clear from Proposition 2.9.

3 Distance-regular graphs with classical parameters

In section 4, we will apply our result in section 2 to the distance-regular graphs with classical parameters. In this section, we study some basic properties of distance-regular graphs with classical parameters that contain no parallelogram of length 2.

Definition 3.1 *A distance-regular graph* Γ *is said to have* **classical parameters** $(D,\ b,\ \alpha,\ \beta)$ *whenever the diameter of* Γ *is* $D \geq 2$*, and the intersection numbers of* Γ *satisfy*

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}(1+\alpha\begin{bmatrix} i-1 \\ 1 \end{bmatrix}) \qquad (0 \leq i \leq D), \tag{46}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha\begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \leq i \leq D), \tag{47}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}. \tag{48}$$

We often use the following formulae.

Lemma 3.2 *Let* Γ *denote a distance-regular graph with classical parameters* $(D,\ b,\ \alpha,\ \beta)$ *and* $D \geq 2$*. Then the following (i)-(ii) hold.*

(i)

$$a_i = \begin{bmatrix} i \\ 1 \end{bmatrix}(\beta - 1 + \alpha(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix})) \quad (1 \leq i \leq D). \tag{49}$$

In particular,

$$a_1 = \beta - 1 + \alpha b\begin{bmatrix} D-1 \\ 1 \end{bmatrix}. \tag{50}$$

(ii)

$$c_2 = (b+1)(\alpha+1). \tag{51}$$

Proof. (i). This is immediate from (8), (46)-(47).

(ii). Set $i = 2$ in (46).

Some basic properties among the constants α, β, b can be easily obtained.

Lemma 3.3 *Let* Γ *denote a distance-regular graph with classical parameters* $(D,\ b,\ \alpha,\ \beta)$ *and* $D \geq 3$*. Then*

(i) b *is an integer, and* $b \neq 0$, $b \neq -1$.

(ii) α, β *are rational numbers, and* $\beta \neq 0$.

(iii) $c_2 \geq b + 1$.

Proof. (i). This is from [1, Proposition 6.2.1].

(ii). Note that c_2, b_0 are integers, and $b_0 \neq 0$. Now (ii) follows by applying (i) to (51), and to (47) with $i = 0$.

(iii). By direct calculation from (46),

$$\begin{aligned} (c_2 - b)(b^2 + b + 1) &= c_3 \\ &> 0. \end{aligned} \tag{52}$$

Since b is an integer,

$$b^2 + b + 1 > 0.$$

Now (iii) follows from (i) and (52).

Recall from Lemma 1.4, a D-bounded distance-regular graph contains no parallelogram of any length. In particular, it contains no parallelogram of length 2.

Lemma 3.4 *([8, Lemma 2.5]) Let* Γ *denote a distance-regular graph with diameter* $D \geq 2$. *Suppose* Γ *contains no parallelogram of length* 2. *Then*

$$a_2 - a_1 c_2 \geq 0. \tag{53}$$

Lemma 3.5 *Let* Γ *denote a distance-regular graph with classical parameters* $(D,\ b,\ \alpha,\ \beta)$ *and* $D \geq 2$. *Then the following (i)-(iii) hold.*

(i)

$$a_2 - a_1 c_2 = (b + 1 - c_2)(b + 1 + a_1). \tag{54}$$

(ii) $c_2 = b + 1$ *if and only if* $\alpha = 0$.

(iii) $a_1 = -b - 1$ *if and only if*

$$\beta = -b(1 + \alpha \begin{bmatrix} D-1 \\ 1 \end{bmatrix}). \tag{55}$$

Proof. (i). This is immediate from (49)-(51).

(ii). Note that $b \neq -1$ by Lemma 3.3(i). Now (ii) is immediate from (51).

(iii). This is immediate from (50).

Lemma 3.6 *Let* Γ *denote a distance-regular graph with classical parameters* $(D,\ b,\ \alpha,\ \beta)$ *and* $D \geq 3$. *Suppose* Γ *contains no parallelogram of length* 2, *and suppose* $c_2 \neq b+1$. *Then*

$$a_1 + b + 1 \leq 0. \tag{56}$$

In particular, $b < -1$.

Proof. By (54), (53), Lemma 3.3(iii),

$$\begin{aligned} a_1 + b + 1 &= \frac{a_2 - a_1 c_2}{b + 1 - c_2} \\ &\leq 0. \end{aligned}$$

This proves (56). Note $b < -1$, since $a_1 \geq 0$ and since $b \neq -1$.

Corollary 3.7 *Let* Γ *denote a distance-regular graph with classical parameters* $(D,\ b,\ \alpha,\ \beta)$ *and* $D \geq 3$. *Suppose* Γ *contains no parallelogram of length* 2. *Then*

Case $a_1 > -b - 1$:

$$c_2 = b + 1, \quad \alpha = 0, \quad a_2 = a_1 c_2.$$

Case $a_1 = -b - 1$:

$$c_2 > b + 1, \quad \alpha < -1, \quad \beta = -b(1 + \alpha \begin{bmatrix} D-1 \\ 1 \end{bmatrix}), \quad a_2 = a_1 c_2.$$

Case $a_1 < -b - 1$:

$$c_2 > b + 1, \quad \alpha < -1, \quad a_2 > a_1 c_2.$$

Proof. Case $a_1 > -b-1$. This follows from Lemma 3.6, Lemma 3.5(ii), (54).

Cases $a_1 \le -b-1$. Note that $c_2 \ne b+1$, otherwise $c_2 \le -a_1$, a contradiction. Now $c_2 > b+1$ follows from Lemma 3.3(iii), and $\alpha < -1$ follows from (51), since $b < -1$ by Lemma 3.6. The remaining statements are clear from (54) and (55).

4 The poset of a D-bounded distance-regular graph with classical parameters

Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and diameter $D \ge 4$. Suppose Γ is D-bounded. We study the poset $P(\Gamma)$ constructed in section 2. We show each interval in $P(\Gamma)$ is modular, and find either $\alpha = 0$ or else β can be expressed in terms of D, α, β. Theorem 4.2 is our main result. The following Lemma will be used in the proof of Theorem 4.2.

Lemma 4.1 *Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \ge 3$. Assume Γ is D-bounded. Then the following (i)-(iii) hold.*

(i)

$$f_i = \begin{cases} (b+1)\frac{\alpha(b^D+1)+\beta(b-1)-\alpha b^{D-i-1}(b^2+1)}{\alpha(b^D+1)+\beta(b-1)-\alpha b^{D-i-1}(b+1)}, & \text{if } b \ne 1 \\ 2, & \text{if } b = 1 \end{cases} \qquad (1 \le i \le D-1),$$

where f_i is defined in Lemma 2.7.

(ii)

$$\begin{aligned} & (f_i - f_{i+1})(b_{D-i-1} - b_{D-i})(b_{D-i-2} - b_{D-i-1}) \\ = \; & b^{3D-3i-4}(b+1-c_2)(\alpha(b^D+1) + \beta(b-1)) \\ & (1 \le i \le D-2). \end{aligned} \tag{57}$$

(iii)

$$b^{3D-3i-4}(b+1-c_2)(\alpha(b^D+1)+\beta(b-1)) \geq 0 \quad (1 \leq i \leq D-2). \tag{58}$$

Furthermore, for each integer i $(1 \leq i \leq D-2)$, *equality holds in (58) if and only if* $f_i = f_{i+1}$.

Proof. (i)-(ii). These are immediate from (37), (47).

(iii). Note that left hand side of (57) is nonnegative by (36), (40). This proves (iii).

Recall that a D-bounded distance-regular graph contains no parallelogram of any length; in particular, the results of section 3 hold.

Theorem 4.2 *Let* Γ *denote a distance-regular graph with classical parameters* $(D,\ b,\ \alpha,\ \beta)$ *and* $D \geq 4$. *Suppose* Γ *is* D*-bounded. Let* I *be an interval of* $P(\Gamma)$ *of rank* r. *Then referring the cases of Corollary 3.7,*

Case $a_1 > -b-1$: *I is isomorphic to* $L_b(r)$.

Case $a_1 \leq -b-1$:

(i)

$$\beta = \alpha \frac{1+b^D}{1-b}. \tag{59}$$

(ii) *I is isomorphic to* $L_{b^2}(r)$. *In particular,* $-b$ *is a prime power.*

Proof. Without loss of generality, we assume $I = [0, x]$, where $\text{rank}(x) = D$. Now we consider the two cases separately.

Case $a_1 > -b-1$. Observe that since $\alpha = 0$, $f_i = b+1 \quad (1 \leq i \leq D-1)$ by Lemma 4.1(i), and I is a modular atomic lattice by Lemma 4.1, Proposition 2.10, Corollary 2.8. Hence I is isomorphic to $L_b(D)$ by [2, Theorem 3.4.1].

Case $a_1 \leq -b-1$. Recall that $b+1-c_2 < 0$ by Corollary 3.7, and $b < -1$ by Lemma 3.6. By considering $i = D-3$ and $i = D-2$ in (58), we get

$$\alpha(b^D+1)+\beta(b-1) \geq 0, \tag{60}$$

and

$$\alpha(b^D + 1) + \beta(b - 1) \leq 0. \tag{61}$$

It follows from (60)-(61),

$$\alpha(b^D + 1) + \beta(b - 1) = 0. \tag{62}$$

Now (59) follows from (62), since $b \neq 1$. Applying (62) to Lemma 4.1(i), we get

$$f_i = b^2 + 1 \quad (1 \leq i \leq D - 1),$$

and I is a modular atomic lattice by Lemma 4.1, Proposition 2.10, Corollary 2.8. Hence I is isomorphic to $L_{b^2}(D)$ by [2, Theorem 3.4.1]. Note that b^2 is a prime power. Recall $b < -1$ is an integer. Hence $-b$ is a prime power.

We now look at each case of Corollary 3.7 under the assumption that Γ is D-bounded. For Case $a_1 > -b - 1$, we quote the result below.

Theorem 4.3 *([1, p195, p277]) Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 3$. Assume Γ is D-bounded and that $a_1 > -b - 1$. Then the following (i)-(iv) hold.*

(i) Suppose $a_1 \neq 0$, $b = 1$. Then Γ is a Hamming graph.

(ii) Suppose $a_1 \neq 0$, $b > 1$. Then Γ is a dual polar graph.

(iii) Suppose $a_1 = 0$, $b = 1$. Then Γ is the Hamming graph $H(D, 2)$.

(iv) Suppose $D \geq 4$, $a_1 = 0$, $b > 1$. Then Γ is the dual polar graph $D_D(b)$.

(See [1, p261, p274] for definition of **Hamming graphs** and **dual polar graphs**.)

Proof. (i)-(ii). Note that $\alpha = 0$ by Corollary 3.7, and Γ contains no parallelogram of length 2 by Lemma 1.4. Now the results follow by [1, Theorem 9.4.4].

(iii). This is a special case of [1, Theorem 6.1.1].

(iv). We have $\beta = 1$ by (50). Hence by [1, Proposition 6.3.1], Γ is bipartite. Γ is clearly 2-bounded; now the result follows by [1, Theorem 9.4.5]

Now we consider the case $a_1 = -b - 1$ of Corollary 3.7.

Theorem 4.4 *Let $\Gamma = (X, R)$ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 4$. Assume Γ is D-bounded and assume $a_1 = -b-1$. Then*

$$\alpha = \frac{b(b-1)}{b+1} \tag{63}$$

and

$$\beta = \frac{-b}{b+1}(b^D + 1). \tag{64}$$

Furthermore, Γ is the dual polar graph ${}^2A_{2D-1}(-b)$.

Proof. By solving (55), (59) for α, β, we get (63)-(64). By (63)-(64), [1, p 194], Γ has the same intersection numbers as ${}^2A_{2D-1}(-b)$. Now Γ is ${}^2A_{2D-1}(-b)$ by [1, Theorem 9.4.7].

Note 4.5 *The dual polar graph ${}^2A_{2D-1}(-b)$ also is a distance-regular graph with classical parameters $(D,\ b^2,\ 0,\ -b)$, so also belongs to case $a_1 > -b-1$ of Corollary 3.7. This is the only class of distance-regular graphs with $D \geq 3$ which has two distinct sets of classical parameters. See [1, p194-197, p274] for definition and more details.*

5 Applications

In this section, we give some applications of Theorem 4.2. Our main results are Corollary 5.3, Corollary 5.7, Theorem 5.8. These applications are based on the following two classifications of D-bounded distance-regular graphs. The first one is a geometric characterization.

Theorem 5.1 *([7, Theorem 7.2]) Let Γ be a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$. Assume $c_2 > 1$, $a_1 \neq 0$. Then Γ is D-bounded if and only if Γ contains no parallelogram of length 2.*

The second classification of D-bounded distance-regular graphs is characterized by parameters.

Theorem 5.2 *Let Γ be a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 3$. Suppose $b < -1$. Then Γ contains no parallelogram of any length. In particular, if in addition, we assume $c_2 > 1$, $a_1 \neq 0$, then Γ is D-bounded.*

Proof. By [6, Theorem 2.12], [8, Lemma 7.3(ii)], Γ contains no parallelogram of any length. If $c_2 > 1$, $a_1 \neq 0$, then Γ is D-bounded by Theorem 5.1.

The following corollary proves Theorem A.

Corollary 5.3 *Let Γ be a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 4$. Suppose $b < -1$, and suppose the intersection numbers $a_1 \neq 0$, $c_2 > 1$. Then*

$$\beta = \alpha \frac{1 + b^D}{1 - b}.$$

Proof. This is immediate from Theorem 5.2 and (59).

In [1, Proposition 6.2.1], Brouwer, Cohen, Neumaier proved essentially the following theorem.

Theorem 5.4 *([1, Proposition 6.2.1]) Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 3$. Then the following (i)-(iii) are equivalent.*

(i) $a_i = a_1 c_i \quad (0 \leq i \leq D)$.

(ii) There exists an integer $i \quad (2 \leq i \leq D)$ such that $a_i = a_1 c_i$.

(iii) $b \in \{c_2 - 1, -a_1 - 1\}$.

In the following theorem, we identify Γ if (i)-(iii) of Theorem 5.4 hold with $b = -a_1 - 1$. This proves Theorem B.

Theorem 5.5 *Let Γ denote a distance-regular graph with diameter $D \geq 4$ and intersection number $c_2 > 1$. Then the following (i)-(ii) are equivalent.*

(i) Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with

$$b = -a_1 - 1.$$

(ii) Γ is the dual polar graph ${}^2A_{2D-1}(-b)$.

Proof. (ii)$\longrightarrow$(i). This is clear.

(i)$\longrightarrow$(ii). Note that $a_1 \neq 0$, since $b \neq -1$ by Lemma 3.3(i). Hence $b < -1$. Now Γ is D-bounded by Theorem 5.2. (ii) is immediate from Theorem 4.4.

In [1, Theorem 8.5.1], Brouwer, Cohen, Neumaier proved the following theorem.

Theorem 5.6 *([1, Theorem 8.5.1]) A regular near polygon Γ of diameter $D \geq 3$ with thick lines is Q-polynomial if and only if it has classical parameters.*

Below, we strengthen this theorem by identifying Γ under assumptions $D \geq 4$, $c_2 > 1$. This proves Theorem C.

Corollary 5.7 *Let Γ denote a Q-polynomial distance-regular graph with diameter $D \geq 4$. Assume the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Suppose Γ is a near polygon graph. Then Γ is a dual polar graph or a Hamming graph.*

Proof. Note that Γ has classical parameters by Theorem 5.6. By [1, Theorem 6.4.1], Γ contains no parallelogram of length 2 and $a_2 = a_1 c_2$. Hence Γ is D-bounded by Theorem 5.1, and Γ is in the first two cases of Corollary 3.7. Now the corollary follows from Theorem 4.3(i)-(ii), Theorem 4.4.

The following theorem proves Theorem D.

Theorem 5.8 *Let Γ denote a distance-regular graph with diameter $D \geq 4$, and the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Then the following (i)-(ii) are equivalent.*

(i) *Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$ with*

$$b = -a_1 - 2. \tag{65}$$

(ii) *Γ is the Hermitian forms graph $Her_{-b}(D)$.*

(See [1, p285] for definition of **Hermitian forms graphs**).

Proof. Note that by [3], [4], [6], Γ is a Hermitian forms graph if and only if

$$\alpha = b - 1, \tag{66}$$

and

$$\beta = -b^D - 1. \tag{67}$$

(i)$\longrightarrow$(ii). Note that $b < -1$. Hence Γ is D-bounded by Theorem 5.2. Now solve (59), (65) for α, β by using (50). We get (66)-(67). This proves (i).

(ii)$\longrightarrow$(i). By (66)-(67), (50), we have (65).

6 Distance-regular graphs with classical parameters and $c_2 = 1$

Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 4$. In previous section, due to the limit in the classification of D-bounded distance-regular graph, we excluded the case $c_2 = 1$. In this section, we obtain some information about the case $c_2 = 1$ by using Hiroshi Suzuki's recent classification of D-bounded distance-regular graphs with $c_2 = 1$, $a_2 > a_1 > 1$. Corollary 6.4 is the main result of this section.

Theorem 6.1 *([5]) Let Γ denote a distance-regular graph with diameter $D \geq 2$ and intersection numbers $c_2 = 1$, $a_2 > a_1 > 1$. Then Γ is D-bounded if and only if Γ contains no parallelogram of any length.*

Lemma 6.2 *Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 3$. Suppose the intersection number $c_2 = 1$. Then Γ contains no parallelogram of any length.*

Proof. Note that Γ contains no parallelogram of length 2, since $c_2 = 1$. Then Γ contains no parallelogram of any length by [6, Theorem 2.12] and by [7, Theorem 7.3(ii)].

Corollary 6.3 *Let Γ denote a distance-regular graph with classical parameters $(D,\ b,\ \alpha,\ \beta)$ and $D \geq 3$. Suppose the intersection numbers $c_2 = 1$, $a_2 > a_1 > 1$. Then Γ is D-bounded.*

Proof. This is immediate from Lemma 6.2, Theorem 6.1.

The following corollary proves Theorem E.

Corollary 6.4 *There is no distance-regular graph Γ with classical parameters $(D,\ b,\ \alpha,\ \beta)$, $D \geq 4$, $c_2 = 1$, and $a_2 > a_1 > 1$.*

Proof. Suppose such Γ exists. By Corollary 6.3, Γ is D-bounded. Hence by (51), Lemma 3.3(ii),

$$\begin{aligned} \alpha &= \frac{c_2}{b+1} - 1 \\ &= \frac{-b}{b+1} \\ &\neq 0, \end{aligned} \tag{68}$$

which belongs to cases $a_1 \leq -b-1$ of Corollary 3.7. Hence by (59),

$$\beta = \alpha \frac{1+b^D}{1-b}. \tag{69}$$

Now by (50), (68)-(69),

$$a_1 = \frac{1}{b-1}.$$

But by Lemma 3.6(i), we have $b < -1$. Hence $a_1 < 0$, a contradiction.

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