Subgraphs Graph in a Distance-regular Graph

Chih-wen Weng*

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Abstract

Let Γ denote a D-bounded distance-regular graph, where $D \geq 3$ is the diameter of Γ. For $0 \leq s \leq D-3$ and a weak-geodetically closed subgraph Δ of Γ with diameter s, define a graph $G(\Delta)$ whose vertex set is the collection of all weak-geodetically closed subgraphs of diameter $s+2$ containing Δ, and vertex Ω is adjacent to vertex Ω' in G if and only if $\Omega \cap \Omega'$ as a subgraph of Γ has diameter $s+1$. We show G is strongly regular and determine its parameters. Furthermore assume D is at least 4 and set $q := b_{D-1}/(b_{D-2} - b_{D-1})$ in the expression of intersection numbers of Γ. We show that $G(\Delta)$ is $J_q(D-s, 2)$, the Johnson graph or its q-analogue, for each weak-geodetically closed subgraph Δ of Γ with diameter s at most $D-3$ if and only if $(b_{s+1} - b_{s+2})/(b_s - b_{s+1}) = q$ for $0 \leq s \leq D-3$, and in this case q is either 1 or a fixed power of a fixed prime.

Keywords: D-bounded distance-regular graph, weak-geodetically closed subgraph, Johnson graph, Grassmann graph.

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*Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C., research partially supported by the NSC grant 95-2115-M-009-002 of Taiwan R.O.C.

1 Introduction

Let Γ denote a distance-regular graph with diameter $D \geq 3$. A sequence of vertices x, y, z of Γ is *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1,$$

where ∂ is the distance function of Γ. A subgraph Δ of Γ is *weak-geodetically closed* whenever for all weak-geodetic sequences of vertices x, y, z of Γ we have

$$x, z \in \Omega \Longrightarrow y \in \Omega.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [8]. We refer the reader to [10], [2], [7], [9], [11], [6] for the constructions of weak-geodetically closed subgraphs of Γ. It is immediate from the definition that a weak-geodetically closed subgraph Δ is an induced subgraph of Γ and the distance function on Δ is induced from that on Γ. Γ is *D-bounded* if (i) all of the weak-geodetically closed subgraphs of Γ are regular; and (ii) for all vertices x, y of Γ, x, y are contained in a common weak-geodetically closed subgraph $\Delta(x, y)$ of diameter $\partial(x, y)$. In fact $\Delta(x, y)$ is uniquely determined by the vertices x and y [11, Corollary 5.4], and is distance-regular [11, Corollary 5.3]. The classification of D-bounded distance-regular graphs with some additional assumptions can be found in [12], [13].

Throughout the paper, let Γ denote a D-bounded distance-regular graph with intersection numbers b_i, c_i for $0 \leq i \leq D$. Note that $b_i > b_{i+1}$ for $0 \leq i \leq D-1$ [12, Lemma 2.6]. Fix an integer $0 \leq s \leq D-3$ and a weak-geodetically closed subgraph Δ of Γ with diameter s. Let $P = P(\Delta)$ denote the collection of weak-geodetically closed subgraphs containing Δ. If $\Delta = \{x\}$ for some vertex x of Γ then we write $P(x)$ for $P(\Delta)$. It was shown that P is a ranked atomic lattice [5], where $\text{rank}(\Omega)$ is $\text{diameter}(\Omega) - s$ for $\Omega \in P$. Let $P_j = P_j(\Delta)$ denote the set of rank j elements in P for $0 \leq j \leq D-s$. For each $1 \leq i \leq D-s$ we define a graph $G(\Delta, i)$ whose vertex set is P_i, and vertex Ω is adjacent to vertex Ω' in $G(\Delta, i)$ if and only if $\Omega \cap \Omega' \in P_{i-1}$, where $\Omega, \Omega' \in P_i$. Our first main result is

Theorem 1.1. *$G(\Delta, 2)$ is either a clique or a strongly regular graph with* *parameters*

$$k = \frac{b_{s+2}(b_s - b_{s+2})}{(b_s - b_{s+1})(b_{s+1} - b_{s+2})}, \tag{1.1}$$

$$\lambda = (\frac{b_{s+1} - b_{s+2}}{b_s - b_{s+1}})^2 + \frac{b_{s+2}}{b_{s+1} - b_{s+2}} - 1, \tag{1.2}$$

$$\mu = (\frac{b_s - b_{s+2}}{b_s - b_{s+1}})^2. \tag{1.3}$$

Theorem 1.1 is a generalization of [4], which proves in the case $\Delta = \{x\}$ for some vertex x of Γ and some additional assumptions. Let $J_1(n, e)$ denote the Johnson graph of the e-sets in a fixed n-element set, and let $J_q(n, e)$ denote its q-analogue, the Grassmann graph of the e-spaces of an n-dimensional vector space over a finite field of q elements. Let $L_1(n)$ denote the lattice consisting of all subsets of a fixed n-element set and let $L_q(n)$ denote its q-analogue, the lattices consisting of all subspaces of a fixed n-dimensional vector space over a finite field of q elements. We study the strongly regular graph $G(\Delta, 2)$ in Theorem 1.1 and find

Theorem 1.2. *As the assumption above, assume Γ has diameter D at least 4 and set*

$$q := \frac{b_{D-1}}{b_{D-2} - b_{D-1}}. \tag{1.4}$$

Then the following (i)-(iv) are equivalent.

(i) q is 1 or a power of a prime, and for each weak-geodetically closed subgraph Δ of diameter s in Γ, $G(\Delta, 2)$ is the distance-regular graph $J_q(D - s, 2)$, where $0 \le s \le D - 3$.

(ii) q is 1 or a power of a prime, and for each weak-geodetically closed subgraph Δ of diameter s in Γ, $G(\Delta, i)$ is the distance-regular graph $J_q(D - s, i)$, where $0 \le s \le D - 3$ and $0 \le i \le D - s$.

(iii) q is 1 or a power of a prime, and $P(x)$ is isomorphic to the lattice $L_q(D)$ for any vertex x of Γ.

(iv)

$$\frac{b_{s+1} - b_{s+2}}{b_s - b_{s+1}} = q \qquad \text{for} \quad 0 \le s \le D - 3. \tag{1.5}$$

The paper is organized as follows. In Section 2 we review the definition of distance-regular graphs and the examples of distance-regular graphs which are used in this paper. Theorem 1.1 is proved in Section 3, and Theorem 1.2 is proved in Section 4.

2 Preliminaries

In this section, we review some definitions and basic concepts. See the book of Brouwer, Cohen, and Neumaier[2] for more background information.

Assume Γ is a simple connected graph with diameter D. For all vertices x in Γ and for $0 \leq i \leq D$, set

$$\Gamma_i(x) := \{y \in \Gamma \mid \partial(x, y) = i\}.$$

Γ is said to be *distance-regular* whenever for $0 \leq h, i, j \leq D$ and for vertices x, y in Γ with $\partial(x, y) = h$, the number

$$p^h_{ij} := \left|\Gamma_i(x) \cap \Gamma_j(y)\right|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1i-1}, a_i := p^i_{1i}, b_i := p^i_{1i+1}$ and $k_i := p^0_{ii}$. Note that $c_1 = 1$, $a_0 = 0$, $b_D = 0$ and

$$k_1 = c_i + a_i + b_i \quad \text{for} \quad 0 \leq i \leq D.$$

A *strongly regular graph* is a distance-regular graph of diameter 2, and in this case we use k, λ, μ to replace b_0, a_1, c_2 respectively.

Let X be a finite set of n elements. The *Johnson graph* $J(n, e)$ *of the e-sets in* X has vertex set

$$\binom{X}{e},$$

the collection of e-subsets of X. Two vertices γ, δ are adjacent whenever $|\gamma \cap \delta| = e - 1$. Sometimes we use the notation $J_1(n, e)$ for $J(n, e)$.

Let F_q denote a field of q elements, and let V denote an n-dimensional vector space over F_q. The *Grassmann graph* $J_q(n, e)$ *of the e-spaces of* V has vertex set

$$\left[\begin{array}{c} V \\ e \end{array} \right],$$

the set of vector subspaces of V of dimension e. Two vertices γ, δ are adjacent whenever $\gamma \cap \delta$ has dimension $e - 1$.

Set

$$\left[\begin{array}{c} i \\ 1 \end{array} \right] := 1 + q + \cdots + q^{i-1}. \tag{2.1}$$

It is well known $J_q(n, e)$ is a distance-regular graph with diameter $D := \min(e, n - e)$ and intersection numbers

$$b_j = q^{2j+1} \left[\begin{array}{c} e-j \\ 1 \end{array} \right] \left[\begin{array}{c} n-e-j \\ 1 \end{array} \right], \tag{2.2}$$

$$c_j = \left[\begin{array}{c} j \\ 1 \end{array} \right]^2 \tag{2.3}$$

for $0 \leq j \leq D$[1, Theorem 9.1.2, Theorem 9.3.3].

3 Strongly Regular Graphs

We prove Theorem 1.1 in this section. Before doing this, we recall a few properties in a D-bounded distance-regular graph. Let Γ denote a D-bounded distance-regular graph where $D \geq 3$ is the diameter of Γ. Let a_i, b_i, c_i denote the intersection numbers of Γ for $0 \leq i \leq D$. Let Δ denote a weak-geodetically closed subgraph of diameter s for $0 \leq s \leq D$. Note that Δ is regular by the assumption (i) of D-bounded definition. In fact Δ is distance-regular with intersection numbers

$$\begin{aligned} a_i(\Delta) &= a_i(\Gamma) \\ c_i(\Delta) &= c_i(\Gamma) \\ b_i(\Delta) &= b_i(\Gamma) - b_s(\Gamma) \end{aligned}$$

for $0 \leq i \leq s$ [11, Corollary 5.3]. In particular a weak-geodetically closed subgraph of diameter 1 is a clique of size $b_0 - b_1 + 1$, and we refer such a clique to a *line*. The intersection of weak-geodetically closed subgraphs is either an empty set or a weak-geodetically closed subgraph. Hence $|\Delta \cap \ell| \in \{0, 1, b_0 - b_1 + 1\}$ for any line ℓ in Γ. Let x denote a vertex in Δ. Then $\Delta_1(x)$ is a disjoint union of $(b_0 - b_s)/(b_0 - b_1)$ cliques of the form $\ell - \{x\}$, where $\ell \subseteq \Delta$ is a line containing x. There are

$$\frac{b_0}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}$$

lines ℓ' containing x such that $\ell' \not\subseteq \Delta$. For such a line ℓ', there exists a unique weak-geodetically closed subgraph Δ' of diameter $s+1$ containing Δ and ℓ'. There are

$$\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}$$

lines ℓ'' (including ℓ') containing x such that $\ell'' - \{x\} \subseteq \Delta' - \Delta$.

Proof of Theorem 1.1.

Fix $x \in \Delta$ and $\Omega \in P_2$. Then $x \in \Delta \subseteq \Omega$ by the construction. First we prove the number $k = k(\Omega)$ as expressed in (1.1). We do this by counting the triples (Ω', ℓ, ℓ') in the order and its reversed order where $\Omega' \in P_2$ such that $\Omega \cap \Omega' \in P_1$, and $\ell, \ell' \subseteq \Omega'$ are lines containing x such that $\ell - \{x\} \subseteq \Omega \cap \Omega' - \Delta$ and $\ell' - \{x\} \subseteq \Omega' - \Omega$. We find

$$\begin{aligned} & k \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \\ = \quad & (\frac{b_0}{b_0 - b_1} - \frac{b_0 - b_{s+2}}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times 1 \end{aligned}$$

to obtain (1.1).

Second we fix another $\Omega' \in P_2$ such that $\Omega \cap \Omega' \in P_1$. We prove the number $\lambda = \lambda(\Omega, \Omega')$ as expressed in (1.2). Let λ_1 (resp. λ_2) denote the number of $\Omega'' \in P_2$ such that

$$\Omega'' \cap \Omega = \Omega'' \cap \Omega' = \Omega' \cap \Omega \tag{3.1}$$

(resp.

$$\Omega'' \cap \Omega \in P_1, \Omega'' \cap \Omega' \in P_1, \Omega \cap \Omega' \cap \Omega'' = \Delta). \tag{3.2}$$

Note that

$$\lambda = \lambda_1 + \lambda_2. \tag{3.3}$$

To determine λ_1 we count the pairs (Ω'', ℓ'') in the order and its reversed order, where $\Omega'' \in P_2$ satisfies (3.1) and $\ell'' \subseteq \Omega''$ is a line such that $\ell'' \not\subseteq \Omega \cup \Omega'$. We find

$$\begin{aligned} & \lambda_1 \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \\ = \quad & (\frac{b_0}{b_0 - b_1} - 2\frac{b_0 - b_{s+2}}{b_0 - b_1} + \frac{b_0 - b_{s+1}}{b_0 - b_1}) \times 1. \end{aligned} \tag{3.4}$$

To determine λ_2 we count the triples (Ω'', ℓ, ℓ') in the order and its reversed order, where $\Omega'' \in P_2$ satisfies (3.2), and $\ell, \ell' \subseteq \Omega''$ are lines containing x such that $\ell - \{x\} \subseteq \Omega - \Omega'$ and $\ell' - \{x\} \subseteq \Omega' - \Omega$. We find

$$\begin{aligned} & \lambda_2 \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times ((\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \\ = \quad & (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_{s+1}}{b_0 - b_1}) \times 1. \end{aligned} \tag{3.5}$$

(1.2) is immediate by solving (3.3)-(3.5) for λ.

Third we fix $\Omega'' \in P_2$ such that $\Omega \cap \Omega'' = \Delta$. We prove the number $\mu = \mu(\Omega, \Omega'')$ as expressed in (1.3). We do this by counting the triples $(\Omega''', \ell, \ell'')$ in the order and its reversed order, where $\Omega''' \in P_2$ such that $\Omega''' \cap \Omega, \Omega''' \cap \Omega'' \in P_1$, and $\ell, \ell'' \subseteq \Omega'''$ are lines containing x such that $\ell - \{x\} \subseteq \Omega - \Omega''$ and $\ell'' - \{x\} \subseteq \Omega'' - \Omega$. We find

$$\begin{aligned} & \mu \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times (\frac{b_0 - b_{s+1}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \\ = \quad & (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times (\frac{b_0 - b_{s+2}}{b_0 - b_1} - \frac{b_0 - b_s}{b_0 - b_1}) \times 1. \end{aligned} \tag{3.6}$$

(1.3) follows from (3.6). □

4 Modularity

Let Γ denote a D-bounded distance-regular graph where $D \geq 4$ is the diameter of Γ. Fix a vertex x of Γ and let $P(x)$ denote the set of weak-geodetically closed subgraphs containing x. We order $P(x)$ by inclusion. We quote a theorem which will be used the proof of Theorem 1.2.

Theorem 4.1. *[12, Proposition 2.10] The following are equivalent*

(i) $\dfrac{b_s - b_{s+2}}{b_s - b_{s+1}}$ *is a constant for any* $0 \leq s \leq D-2$*;*

(ii) every rank 3 *interval in* $P(x)$ *is modular;*

(iii) every interval in $P(x)$ *is modular.*

Proof of Theorem 1.2. ((iv)$\Longrightarrow$(iii)) Referring to (1.4) and (1.5) we find the expression in Theorem 4.1(i) is $q+1$. Hence by Theorem 4.1(iii) every interval in $P(x)$ is modular. Then (iii) follows by [3, Theorem 3.4.1]. Here we need the assumption $D \geq 4$.

((iii)$\Longrightarrow$(ii)) Suppose $P(x)$ is isomorphic to $L_q(D)$ and Δ is a weak-geodetically closed subgraph of diameter s containing the vertex x in Γ. Then $P(\Delta)$ is isomorphic to $L_q(D-s)$ and (ii) follows from the definition of $J_q(D-s, i)$.

((ii)$\Longrightarrow$(i)) This is clear.

((i)$\Longrightarrow$(iv)) Applying (2.3) to (1.3) , we find

$$\frac{b_s - b_{s+2}}{b_s - b_{s+1}} = q + 1 \tag{4.1}$$

for $0 \leq s \leq D-3$. (iv) follows by subtracting 1 in both sides of (4.1). □

Conjecture 4.2. Every D-bounded distance-regular graph with $D \geq 4$ satisfies the the equivalence conditions (i)-(iv) in Theorem 1.2.

Remark 4.3. Referring to the notations in [1, Section 8.4], let Γ denote a distance-regular with classical parameters (D, b, α, β). Then the Q-sequence $(\sigma_0, \sigma_1, \ldots, \sigma_D)$ of Γ satisfies

$$\frac{\sigma_{s+1} - \sigma_{s+2}}{\sigma_s - \sigma_{s+1}} = b^{-1} \tag{4.2}$$

for $0 \leq s \leq D-2$. The sequence $(b_0, b_1, \ldots, b_D)$ in Theorem 1.2(iv) looks like a resemblance of $(\sigma_0, \sigma_1, \ldots, \sigma_D)$ in its combinatorial part.

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