Cycle-Symmetric Matrices

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Abstract. A real square matrix is *cycle-symmetric*, if any two of its entries in the symmetric positions have the same sign, and if the product of the entries in a cycle is the same as that in the reverse cycle. We characterize cycle-symmetric matrices to be matrices which are similar to symmetric matrices by real diagonal matrices. A few applications are given.

Keywords: Cycle-symmetricity; Eigenvalues; Interlacing; Rank

1. Introduction.

In [1], B. Fiedler and T. Gedeon study a class of systems of differential equations in $\mathbb{R}^n$ which exhibits convergent dynamics. There are two assumptions on entries of the matrices related to the systems of equations. We shall call them the *cycle-symmetric* which will be formally defined below. The cycle-symmetric property is a natural generalization of the symmetric property and of the tree-patterned property on real square matrices which are studied by many authors in algebraic and combinatorial matrix theory. In this paper, we show that a cycle-symmetric matrix is similar to a symmetric matrix by an invertible real diagonal matrix. A consequence is that a cycle-symmetric matrix is diagonalizable with all real eigenvalues, and has the interlacing property. An application to minimum rank of matrices with prescribed graph a tree is given.

Throughout this paper, let A denote an n by n matrix with entries $a_{ij} \in \mathbb{R}$ $\quad (i, j \in \{1, 2, \cdots, n\})$. A is called *sign-symmetric* if for all integers $i, j \in \{1, \cdots, n\}$, $a_{ij}a_{ji} > 0$ when $a_{ij} \neq 0$. It is immediate from the definition that

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if A is sign-symmetric, then for each pair of indices i, j, $a_{ij} = 0$ if and only if $a_{ji} = 0$. If A is sign-symmetric, we define the graph $\Gamma(A)$ having n vertices labeled $1, 2, \cdots, n$. For $i \neq j$, the unordered pair (i, j) will be an edge in $\Gamma(A)$ if and only if $a_{ij} \neq 0$. We refer the reader to [3], [4], [5], [6] for the discussion on the matrix A which is assumed to be symmetric and $\Gamma(A)$ a tree.

Let G denote an undirected graph without loops, or multiple edges. By a *walk* of length t ended with the vertex v, we mean a sequence of vertices $u_0 u_1 \cdots u_t$ such that $t \geq 1$, $u_t = v$ and for each $i \in \{0, \cdots, t-1\}$, $u_i u_{i+1}$ is an edge in G. By a *path*, we mean a walk without repeated vertices except the first and the last could be equal. By a *closed walk*, we mean a walk with the same first and last vertex. By a *cycle*, we mean a closed walk to be a path of length greater than or equal to 3.

An n by n matrix A is called *cycle-symmetric* if A is sign-symmetric and

$$a_{u_0 u_1} a_{u_1 u_2} \cdots a_{u_{t-1} u_t} = a_{u_1 u_0} a_{u_2 u_1} \cdots a_{u_t u_{t-1}}, \tag{1.1}$$

for any cycles $u_0 u_1 \cdots u_t$ in the complete graph of the vertex set $\{1, \cdots, n\}$. Observe that it makes no effects on the definition if the cycles in the complete graph are restricted to the cycles in $\Gamma(A)$. It is clear from the definition that a symmetric matrix is cycle-symmetric. Observe that a tree contains no cycles, hence we have the following result.

Proposition 1.1. Let A denote a sign-symmetric matrix. Assume $\Gamma(A)$ is a tree. Then A is cycle-symmetric.

In particular, a tridiagonal sign-symmetric matrix is cycle-symmetric. We say A is *similar* to B by P if P is invertible and $B = P^{-1}AP$. A is *diagonalizable* if A is similar to a diagonal matrix by some invertible matrix P. It is well known that a symmetric matrix is diagonalizable and so is a tridiagonal sign-symmetric matrix. We shall prove in next section that this is also true for a cycle-symmetric matrix. Fix a subset $\alpha \subseteq \{1, \cdots, n\}$ of cardinality s, and let $A[\alpha]$ denote the submatrix of A obtained by deleting the rows and columns not indexed in α. $A[\alpha]$ is called the *principle submatrix* of A indexed by α. Observe that a principle submatrix of a cycle-symmetric matrix is cycle-symmetric. Suppose for this moment, A has n real eigenvalues $\theta_1, \cdots, \theta_n$, and $A[\alpha]$ has s real eigenvalues η_1, $\cdots$, η_s. We say the eigenvalues of $A[\alpha]$ *interlace* the eigenvalues of A if

$$\theta_j \geq \eta_j \geq \theta_{n-s+j} \quad (1 \leq j \leq s).$$

We say A has the *interlacing* property if the eigenvalues of $A[\alpha]$ interlace the eigenvalues of A for all $\alpha \subseteq \{1, \cdots, n\}$. It is a classical result that symmetric matrices have the interlacing property. See [2, p85] for recent reference. We shall also prove that cycle-symmetric matrices have the interlacing property.

Given a graph G on n vertices, we define the number $m(G)$ by

$$m(G) := \min\{\text{rank } A \mid \Gamma(A) = G, \ \text{ where } \ A \ \text{ is } \ \text{symmetric}\}, \tag{1.2}$$

and the number $m^*(G)$ by

$$m^*(G) := \min\{\text{rank } A \mid \Gamma(A) = G, \ \text{ where } \ A \ \text{ is } \ \text{sign}-\text{symmetric}\}. \tag{1.3}$$

In [4], P. Nylen gives an algorithm to compute $m(T)$, for a tree T. We improve the algorithm in [6]. We shall prove in this paper $m(T) = m^*(T)$ for all trees T.

2. The Characterization of Cycle-Symmetric Matrices.

The following lemma will be used to prove our main theorem.

Lemma 2.1. Let G denote a graph. Then there exists a vertex u in G such that either u has degree less than or equal to one, or for any distinct neighbors v, w of u, there exists a cycle containing vuw.

Proof. We prove the lemma by induction on the number of edges in G. It is clear if G has no edges or even if G has no cycles. Assume G has a cycle. Delete an edge e from a cycle C in G and form a new graph H. By induction, pick a vertex u in H which satisfies the lemma. Assume u has degree at least 2 in G, otherwise we are done. Pick any two neighbors v, w of u in G. If v, w both are neighbors of u in H then we can find a cycle in H (so in G) containing vuw. So we can assume one of v, w, say v, is not a neighbor of u in H. That is, $vu = e$. Suppose uw' is the other edge in C. Then $C = vuw'W$ for a path W not containing u and ended with v. If $w' = w$ then C is the cycle containing vuw. Suppose $w' \neq w$. Then by construction, we can find a cycle $C' = w'uwW'$ in H containing $w'uw$, where W' is a path not containing u and ended with w'. Now $vuwW'W$ is a closed walk and the vertex u does not appear in W and W'. Then by traveling along the closed walk $vuwW'W$ and by removing extra cycles, we find a cycle containing vuw. This proves the lemma.

Theorem 2.2. Let A denote an n by n real matrix. Then A is cycle-symmetric if and only if there exists an invertible real diagonal matrix D such that $D^{-1}AD$ is symmetric.

Proof. ($\Longleftarrow$) Suppose $D = \text{diag}\{d_1, \cdots, d_n\}$ is invertible and $D^{-1}AD$ is symmetric. Comparing the entries in $D^{-1}AD$,

$$\frac{a_{ij}d_j}{d_i} = \frac{a_{ji}d_i}{d_j} \qquad \text{for all } \ i, j \in \{1, \cdots n\}. \tag{2.1}$$

It is immediate from (2.1) that A is sign-symmetric. To prove A is cycle-symmetric, assume $u_0u_1 \cdots u_t$ is a cycle in $\Gamma(A)$. Then by (2.1),

$$\begin{aligned} & a_{u_0u_1}a_{u_1u_2} \cdots a_{u_{t-1}u_t} \\ =& a_{u_1u_0}\frac{d_0^2}{d_1^2}a_{u_2u_1}\frac{d_1^2}{d_2^2} \cdots a_{u_tu_{t-1}}\frac{d_{t-1}^2}{d_t^2} \\ =& a_{u_1u_0}a_{u_2u_1} \cdots a_{u_tu_{t-1}}. \end{aligned}$$

This proves A is cycle-symmetric.

($\Longrightarrow$) We prove by induction on n. $n = 1$ is clear. We pick a vertex u in $\Gamma(A)$ satisfying Lemma 2.1. Write $\alpha = \{1, \cdots, n\} \setminus \{u\}$. By induction, choose an invertible diagonal matrix $D_1 = \text{diag}\{d_1, \cdots, d_{u-1}, d_{u+1}, \cdots, d_n\}$ such that $D_1^{-1}A[\alpha]D_1$ is symmetric. By viewing $D_1^{-1}A[\alpha]D_1$ as indexed by the set α, and comparing its entries,

$$\frac{a_{ij}d_j}{d_i} = \frac{a_{ji}d_i}{d_j} \qquad \text{for all } i, j \in \alpha. \tag{2.2}$$

If $a_{uj} = 0$ for all $j \in \alpha$, then we assign $d_u = 1$, otherwise

$$d_u = d_v\sqrt{\frac{a_{uv}}{a_{vu}}}, \tag{2.3}$$

where v is the least integer in α such that $a_{uv} \neq 0$. Now set $D = \text{diag}\{d_1, \cdots, d_n\}$. Since we have known (2.2), to prove $D^{-1}AD$ is symmetric, we only need to show

$$\frac{a_{uw}d_w}{d_u} = \frac{a_{wu}d_u}{d_w} \quad \text{for all } w \in \alpha. \tag{2.4}$$

This is clear if $w = v$ by (2.3), or if $a_{uw} = 0$, so we assume $w \neq v$ and $a_{uw} \neq 0$. Note that v, w are distinct neighbors of u in $\Gamma(A)$. By Lemma 2.1, pick a cycle $u_0u_1u_2 \cdots u_{t-1}u_t$ in $\Gamma(A)$, where $u_0 = u_t = u$, $u_1 = w$, $u_{t-1} = v$ and $u_1, \cdots, u_{t-1} \in \alpha$. Now by A being cycle-symmetric, (2.2) and (2.3),

$$\begin{aligned} \frac{a_{uw}d_w}{d_u} &= \frac{a_{uw}a_{u_1u_2} \cdots a_{u_{t-1}u_t}d_w}{a_{u_1u_2} \cdots a_{u_{t-1}u_t}d_u} \\ &= \frac{a_{wu}a_{u_2u_1} \cdots a_{u_tu_{t-1}}d_w}{a_{u_2u_1}\frac{d_{u_1}^2}{d_{u_2}^2} \cdots a_{u_tu_{t-1}}\frac{d_{u_{t-1}}^2}{d_{u_t}^2}d_u} \\ &= \frac{a_{wu}d_u}{d_w}. \end{aligned}$$

This completes the proof.

Corollary 2.3. Let A denote an n by n real matrix. Suppose A is cycle-symmetric. Then A is diagonalizable and all eigenvalues of A are reals. Furthermore, A has the interlacing property.

Proof. Applying Theorem 2.2, A is diagonalizable and has all eigenvalues real, since A is similar to a symmetric matrix. The interlacing property of A follows from that a symmetric matrix has this property and the observation

$$D^{-1}AD[\alpha] = D^{-1}[\alpha]A[\alpha]D[\alpha],$$

for all invertible diagonal matrices D and all $\alpha \subseteq \{1, \cdots, n\}$.

Corollary 2.4. $m(T) = m^*(T)$ for all trees T.

Proof. $m^*(T) \leq m(T)$ is immediate from the definitions in (1.2)-(1.3). Let A denote a sign-symmetric matrix with $\Gamma(A) = T$ and $\text{rank}(A) = m^*(T)$. Note that A is cycle-symmetric by Proposition 1.1. Choose an invertible real diagonal matrix D such that $D^{-1}AD$ is symmetric. Observe $\Gamma(D^{-1}AD) = T$, since $D^{-1}AD$ and A have the same nonzero entries. Hence

$$\begin{aligned} m(T) &\leq \text{rank}(D^{-1}AD) \\ &= \text{rank}(A) \\ &= m(T^*). \end{aligned}$$

This proves $m(T) = m^*(T)$.

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