# An inequality for regular near polygons

Paul Terwilliger [*] Chih-wen Weng [†]

March 5, 2004

**Abstract**

Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency $k$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let $\theta_1$ denote the second largest eigenvalue of $\Gamma$. We show

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}.$$

We show the following (i)–(iii) are equivalent. (i) Equality is attained above; (ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_1$; (iii) $\Gamma$ is a dual polar graph or a Hamming graph.

Keywords: near polygon, distance-regular graph, $Q$-polynomial, dual polar graph, Hamming graph.

AMS Subject Classification: 05E30.

# 1 Introduction

Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$ (see Section 2 for formal definitions). Suppose the intersection numbers $a_1 > 0$ and $c_2 > 1$. It was shown by Brouwer, Cohen and Neumaier that if $\Gamma$ has classical parameters $(d, q, 0, \beta)$ then $\Gamma$ is a Hamming graph or a dual polar graph [2, Theorem 9.4.4]. The same conclusion was obtained by the second author under the assumption that $\Gamma$ is $Q$-polynomial and has diameter

[*]Department of Mathematics, University of Wisconsin-Madison, USA
[†]Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C.

$d \geq 4$ [11, Corollary 5.7]. Let $\theta_0 > \theta_1 > \cdots > \theta_d$ denote the eigenvalues of $\Gamma$. It is known that $\theta_0 = k$, where $k$ denotes the valency of $\Gamma$. By [2, Proposition 4.4.6(i)],

$$\theta_d \geq -\frac{k}{a_1+1},$$

with equality if and only if $\Gamma$ is a near $2d$-gon. We now state our result.

**Theorem 1.1.** *Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$, valency $k$, and intersection numbers $a_1 > 0$, $c_2 > 1$. Let $\theta_1$ denote the second largest eigenvalue of $\Gamma$. Then*

$$\theta_1 \leq \frac{k-a_1-c_2}{c_2-1}. \tag{1.1}$$

*Moreover, the following (i)–(iii) are equivalent.*

*(i) Equality is attained in (1.1);*

*(ii) $\Gamma$ is $Q$-polynomial with respect to $\theta_1$;*

*(iii) $\Gamma$ is a dual polar graph or a Hamming graph.*

# 2 Preliminaries

In this section we review some definitions and basic concepts. See the books by Bannai and Ito [1] or Brouwer, Cohen, and Neumaier [2] for more background information.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph without loops or multiple edges, with vertex set $X$, edge set $R$, path-length distance function $\partial$ and diameter $d := \max\{\partial(x,y) | x, y \in X\}$. For $x \in X$ and for all integers $i$, set

$$\Gamma_i(x) := \{y | y \in X, \partial(x,y) = i\}.$$

Let $k$ denote a nonnegative integer. We say $\Gamma$ is *regular* with *valency* $k$ whenever $|\Gamma_1(x)| = k$ for all $x \in X$. Pick an integer $i$ $(0 \leq i \leq d)$. For $x \in X$ and for $y \in \Gamma_i(x)$, set

$$B(x,y) := \Gamma_1(x) \cap \Gamma_{i+1}(y), \tag{2.1}$$
$$A(x,y) := \Gamma_1(x) \cap \Gamma_i(y), \tag{2.2}$$
$$C(x,y) := \Gamma_1(x) \cap \Gamma_{i-1}(y). \tag{2.3}$$

The graph $\Gamma$ is said to be *distance-regular* whenever for all integers $i$ $(0 \leq i \leq d)$, and for all $x, y \in X$ with $\partial(x,y) = i$, the numbers

$$c_i := |C(x,y)|, \qquad a_i := |A(x,y)|, \qquad b_i := |B(x,y)| \tag{2.4}$$

are independent of $x$ and $y$. We call the $c_i$, $a_i$, $b_i$ the *intersection numbers* of $\Gamma$. We observe $c_0 = 0$, $a_0 = 0$, $b_d = 0$ and $c_1 = 1$. For the rest of this paper we assume $\Gamma$ is distance-regular with diameter $d \geq 3$. We observe $\Gamma$ is regular with valence $k = b_0$ and that

$$c_i + a_i + b_i = k \qquad (0 \leq i \leq d) \tag{2.5}$$

[2, p. 126].

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq d$ let $A_i$ denote the matrix in $\mathrm{Mat}_X(\mathbb{R})$ which has $xy$ entry

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

We call $A_i$ the *ith distance matrix* of $\Gamma$. Observe (ai) $A_0 = I$; (aii) $\sum_{i=0}^{d} A_i = J$; (aiii) $A_i^t = A_i$ $(0 \leq i \leq d)$, (aiv) there exist constants $p_{ij}^h$ $(0 \leq i, j \leq d)$ such that $A_i A_j = \sum_{h=0}^{d} p_{ij}^h A_h$, where $I$ denotes the identity matrix and $J$ denotes the all ones matrix. We abbreviate $A := A_1$ and call this the *adjacency matrix* of $\Gamma$. Let $\mathbf{M}$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{R})$ generated by $A$. Using (ai)–(aiv) we find $A_0, A_1, \ldots, A_d$ form a basis of $\mathbf{M}$. We call $\mathbf{M}$ the *Bose-Mesner algebra* of $\Gamma$. By [1, p. 59, p. 64], $\mathbf{M}$ has a second basis $E_0$, $E_1$, …, $E_d$ such that (ei) $E_0 = |X|^{-1}J$; (eii) $\sum_{i=0}^{d} E_i = I$; (eiii) $E_i^t = E_i$ $(0 \leq i \leq d)$; (eiv) $E_i E_j = \delta_{ij} E_i$ $(0 \leq i, j \leq d)$. We call $E_0$, $E_1$, …, $E_d$ the *primitive idempotents* for $\Gamma$. Since $E_0$, $E_1$, …, $E_d$ form a basis for $\mathbf{M}$ there exist real scalars $\theta_0, \theta_1, \ldots, \theta_d$ such that $A = \sum_{i=0}^{d} \theta_i E_i$. By this and (eiv) we find $AE_i = \theta_i E_i$ $(0 \leq i \leq d)$. Observe $\theta_0$, $\theta_1$, …, $\theta_d$ are mutually distinct since $A$ generates $\mathbf{M}$. We assume the $E_i$ are indexed so that $\theta_0 > \theta_1 > \cdots > \theta_d$. We call $\theta_i$ the *eigenvalue* of $\Gamma$ corresponding to $E_i$. By [1, p. 197] we have $\theta_0 = k$ and $-k \leq \theta_i \leq k$ $(0 \leq i \leq d)$. We call $\theta_0$ the *trivial* eigenvalue.

Let $\theta$ denote an eigenvalue of $\Gamma$ and let $E$ denote the corresponding primitive idempotent. Since $E \in \mathbf{M}$, there exist real numbers $\sigma_0$, $\sigma_1$, …, $\sigma_d$ such that

$$E = m|X|^{-1} \sum_{i=0}^{d} \sigma_i A_i, \tag{2.6}$$

where $m = \text{rank } E$. We have $\sigma_0 = 1$ and

$$c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i \qquad (0 \le i \le d), \tag{2.7}$$

where $\sigma_{-1}$, $\sigma_{d+1}$ denote indeterminates [1, p. 191]. The sequence $\sigma_0, \sigma_1, \dots, \sigma_d$ is called the *cosine sequence* associated with $\theta$. Let $\sigma_0, \sigma_1, \dots, \sigma_d$ denote the cosine sequence associated with the eigenvalue $k$. Comparing (2.5) and (2.7) we find $\sigma_i = 1$ $(0 \le i \le d)$. By the *trivial cosine sequence* of $\Gamma$ we mean the cosine sequence associated with $k$. Let $\theta$ denote an eigenvalue of $\Gamma$ and let $\sigma_0, \sigma_1, \dots, \sigma_d$ denote the corresponding cosine sequence. By (2.7),

$$\sigma_1 = \theta k^{-1}, \tag{2.8}$$

$$\sigma_2 = \frac{\theta^2 - a_1\theta - k}{kb_1}. \tag{2.9}$$

Combining (2.5), (2.8) and (2.9) we find

$$(\sigma_1 - \sigma_2) b_1 = (\theta + 1)(\sigma_0 - \sigma_1). \tag{2.10}$$

Set $V = \mathbb{R}^X$ (column vectors). We define the inner product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V).$$

For each $x \in X$ set

$$\hat{x} = (0, 0, \cdots, 1, 0, \cdots, 0)^t,$$

where the 1 is in coordinate $x$. We observe $\{\hat{x} |\ x \in X\}$ is an orthonormal basis for $V$. By (2.6), for $x, y \in X$ we have

$$\langle E\hat{x}, E\hat{y} \rangle = m|X|^{-1} \sigma_i, \tag{2.11}$$

where $i = \partial(x, y)$.

By a *clique* in $\Gamma$ we mean a nonempty set consisting of mutually adjacent vertices of $\Gamma$. A given clique in $\Gamma$ is said to be *maximal* whenever it is not properly contained in a clique. The graph $\Gamma$ is said to be a *near polygon* whenever

(i) Each maximal clique has cardinality $a_1 + 2$;

(ii) For all maximal cliques $\ell$ and for all $x \in X$, either

(iia) $\partial(x, y) = d$ for all $y \in \ell$, or

(iib) there exists an integer $i$ $(0 \leq i \leq d-1)$ and a unique $z \in \ell$ such that $\partial(x, z) = i$ and $\partial(x, y) = i + 1$ for all $y \in \ell - \{z\}$.

We give an alternate description of a near polygon. Let $K_{1,2,1}$ denote the graph with 4 vertices $s, x, y, s'$ such that $\partial(s, x) = \partial(s, y) = \partial(x, y) = \partial(x, s') = \partial(y, s') = 1$ and $\partial(s, s') = 2$. Then by [2, Theorem 6.4.1] $\Gamma$ is a near polygon if and only if both the following (i')-(ii') hold.

(i') $\Gamma$ does not contain an induced $K_{1,2,1}$ subgraph;

(ii')

$$a_i = a_1 c_i \qquad (0 \leq i \leq d-1). \tag{2.12}$$

Assume $\Gamma$ is a near polygon. Then

$$a_d \geq a_1 c_d. \tag{2.13}$$

Moreover $a_d = a_1 c_d$ if and only if no maximal clique satisfies (iia) above [2, Theorem 6.4.1]. In this case we call $\Gamma$ a *near* $2d$*-gon*. Otherwise we call $\Gamma$ a *near* $(2d+1)$*-gon*. Assume $\Gamma$ is a near polygon. The Hoffman bound states that

$$\theta_d \geq -\frac{k}{a_1 + 1}, \tag{2.14}$$

with equality if and only if $\Gamma$ is a near $2d$-gon [2, Proposition 4.4.6(i)].

**Definition 2.1.** Let $\Gamma$ denote a distance-regular graph with diameter $d \geq 3$. We say $\Gamma$ has *classical parameters* $(d, q, \alpha, \beta)$ whenever the intersection numbers are given by

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right) \qquad (1 \leq i \leq d), \tag{2.15}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}\right) \qquad (0 \leq i \leq d), \tag{2.16}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + q + q^2 + \cdots + q^{j-1}. \tag{2.17}$$

We give two examples of near polygon distance-regular graphs with classical parameters $(d, q, \alpha, \beta)$.

**Example 2.2. The Hamming graph** $H(d,n)$ $(d \geq 3, n \geq 2)$ **[4], [5], [6], [8].**

$X$ is the set of all $d$-tuples of elements from the set $\{1, 2, \cdots, n\}$,

$xy \in R$ iff $x, y$ differ in exactly 1 coordinate $\quad (x, y \in X)$,

$q = 1, \quad \alpha = 0, \quad \beta = n - 1,$

$c_i = i, \quad b_i = (d-i)(n-1), \quad a_i = (n-2)i \quad (0 \leq i \leq d),$

$\theta_i = (d-i)(n-1) - i \quad (0 \leq i \leq d).$

**Example 2.3. The Dual polar graphs [3], [7].**

Let $U$ denote a finite vector space with one of the following non-degenerate forms:

| name | dim($U$) | field | form | $\epsilon$ |
|---|---|---|---|---|
| $B_d(p^n)$ | $2d+1$ | $GF(p^n)$ | quadratic | 1 |
| $C_d(p^n)$ | $2d$ | $GF(p^n)$ | symplectic | 1 |
| $D_d(p^n)$ | $2d$ | $GF(p^n)$ | quadratic / (Witt index $d$) | 0 |
| ${}^2D_{d+1}(p^n)$ | $2d+2$ | $GF(p^n)$ | quadratic / (Witt index $d$) | 2 |
| ${}^2A_{2d}(p^n)$ | $2d+1$ | $GF(p^{2n})$ | Hermitean | $\frac{3}{2}$ |
| ${}^2A_{2d-1}(p^n)$ | $2d$ | $GF(p^{2n})$ | Hermitean | $\frac{1}{2}$ |

where $d \geq 3$, $p$ is prime and $n \in \mathbb{N} \setminus \{0\}$.

A subspace of $U$ is called *isotropic* whenever the form vanishes completely on that subspace. In each of the above cases, the dimension of any maximal isotropic subspace is $d$.

$X$ is the set all maximal isotropic subspaces of $U$,

$xy \in R$ iff $\dim(x \cap y) = d - 1 \quad (x, y \in X)$,

$\alpha = 0, \quad \beta = q^\epsilon$,

$$c_i = \frac{q^i - 1}{q - 1}, \qquad a_i = \frac{q^{i+\epsilon} - q^i - q^\epsilon + 1}{q - 1} \qquad (0 \le i \le d),$$

$$b_i = \frac{q^{i+\epsilon}(q^{d-i} - 1)}{q - 1} \qquad (0 \le i \le d - 1),$$

$$\theta_i = \frac{q^{d+\epsilon-i} - q^\epsilon - q^i + 1}{q - 1} \qquad (0 \le i \le d),$$

where

$q = p^n, p^n, p^n, p^n, p^{2n}, p^{2n}$ respectively.

Note that the dual polar graphs on $B_d(p^n)$ and $C_d(p^n)$ are isomorphic if and only if $p$ is equal to 2 [2, p. 277].

The following three theorems will be used in the proof of our results.

**Theorem 2.4.** *([9, Theorem 4.1]) Let* $\Gamma$ *denote a distance-regular graph with diameter* $d \ge 3$*, and let* $q$ *denote a real number at least* 1*. Then the following conditions (i), (ii) are equivalent.*

*(i)* $\Gamma$ *has a nontrivial cosine sequence* $\sigma_0, \sigma_1, \ldots, \sigma_d$ *such that* $\sigma_{i-1} - q\sigma_i$ *is independent of* $i \quad (1 \le i \le d)$.

*(ii) The intersection numbers of* $\Gamma$ *are such that* $qc_i - b_i - q(qc_{i-1} - b_{i-1})$ *is independent of* $i \quad (1 \le i \le d)$.

*Furthermore, if (i), (ii) hold, then*

$$c_3 \ge (c_2 - q)(1 + q + q^2). \tag{2.18}$$

**Theorem 2.5.** *([9, Theorem 4.2]) Let* $\Gamma$ *denote a distance-regular graph with diameter* $d \geq 3$*, and let* $q$ *denote a real number at least* 1*. Then the following conditions (i), (ii) are equivalent.*

*(i) Statements (i), (ii) hold in Theorem 2.4, and* $c_3 = (c_2 - q)(1 + q + q^2)$.

*(ii) There exists* $\alpha,\ \beta \in \mathbb{R}$ *such that* $\Gamma$ *has classical parameters* $(d, q, \alpha, \beta)$.

**Theorem 2.6.** *([2, Theorem 9.4.4]) Let* $\Gamma$ *denote a distance-regular graph with diameter* $d \geq 3$ *with classical parameters* $(d, q, 0, \beta)$*. Assume the intersection numbers* $a_1 > 0$ *and* $c_2 > 1$*. Suppose* $\Gamma$ *is a near polygon. Then* $\Gamma$ *is a dual polar graph or a Hamming graph.*

# 3 The inequality

In this section we obtain the inequality in Theorem 1.1.

**Lemma 3.1.** *Let* $\Gamma$ *denote a near polygon distance-regular graph with diameter* $d \geq 3$*, valency* $k$*, and intersection numbers* $a_1 > 0$*,* $c_2 > 1$*. Let* $\theta_1$ *denote the second largest eigenvalue of* $\Gamma$*. Then*

$$\theta_1 \leq \frac{k - a_1 - c_2}{c_2 - 1}. \tag{3.1}$$

*Proof.* Abbreviate $E = E_1$. Let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the cosine sequence associated with $\theta_1$. Fix any two vertices $x, y \in X$ with $\partial(x, y) = 2$. We consider the vectors

$$u = \sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w}, \tag{3.2}$$

$$v = E\hat{x} - E\hat{y}. \tag{3.3}$$

By the Cauchy-Schwartz inequality,

$$\|u\|^2 \|v\|^2 \geq \langle u,\ v\rangle^2. \tag{3.4}$$

We compute the terms in (3.4). Using (2.11), (3.2), (3.3) we find

$$\|v\|^2 = 2m|X|^{-1}(\sigma_0 - \sigma_2), \tag{3.5}$$

$$\langle u, v\rangle = 2ma_2|X|^{-1}(\sigma_1 - \sigma_2). \tag{3.6}$$

We now compute $\|u\|^2$. To do this we first discuss the distances between vertices in $A(x,y)$ and vertices in $A(y,x)$. We claim that for all $z \in A(x,y)$, $z$ is adjacent to $c_2 - 1$ vertices in $A(y,x)$ and is at distance 2 from the remaining $a_2 - c_2 + 1$ vertices in $A(y,x)$. To see this fix $z \in A(x,y)$. Then $\ell := A(x,z) \cup \{x,z\}$ is a maximal clique; hence there exists a unique vertex $s \in \ell$ with $\partial(s,y) = 1$. That is $s \in C(x,y) \cap C(z,y)$. Observe $|C(x,y) \cap C(z,y)| = 1$, since any other $s' \in C(x,y) \cap C(z,y)$ will cause either $xss'y$ or $sxzs'$ to be a $K_{1,2,1}$ subgraph. Hence there are $c_2 - 1$ vertices in $C(z,y) \cap A(y,x)$. Observe for $w \in A(y,x)$ we have $\partial(w,x) = 2$ and $\partial(w,s) \leq 2$ so $\partial(w,z) \leq 2$. We have now proved the claim. Using the claim and applying (2.11) we find

$$\begin{aligned} \|u\|^2 &= \| \sum_{z \in A(x,y)} E\hat{z}\|^2 + \| \sum_{w \in A(y,x)} E\hat{w}\|^2 - 2\langle \sum_{z \in A(x,y)} E\hat{z}, \sum_{w \in A(y,x)} E\hat{w}\rangle \\ &= 2ma_2|X|^{-1}(\sigma_0 + (a_1 - c_2)\sigma_1 + (c_2 - a_1 - 1)\sigma_2). \end{aligned} \tag{3.7}$$

Evaluating (3.4) using (3.5)–(3.7) we routinely find

$$(\sigma_0 + (a_1 - c_2)\sigma_1 + (c_2 - a_1 - 1)\sigma_2)(\sigma_0 - \sigma_2) \geq a_2(\sigma_1 - \sigma_2)^2. \tag{3.8}$$

Evaluating (3.8) using (2.8), (2.9), (2.12) we obtain

$$(\theta_1 - k)^2(\theta_1(a_1 + 1) + k)(k - \theta_1(c_2 - 1) - a_1 - c_2) \geq 0. \tag{3.9}$$

Clearly $(\theta_1 - k)^2 > 0$. By (2.14) and since $\theta_1 > \theta_d$ we find $\theta_1(a_1 + 1) + k > 0$. Evaluating (3.9) using these comments we find

$$k - \theta_1(c_2 - 1) - a_1 - c_2 \geq 0$$

and (3.1) follows. □

**Remark 3.2.** *Referring to Example 2.2 and Example 2.3, the eigenvalue $\theta_1$ satisfies (3.1) with equality.*

We comment on the proof of Lemma 3.1.

**Lemma 3.3.** *With the notation of Lemma 3.1, the following (i)–(iii) are equivalent.*

*(i) Equality is attained in (3.1).*

*(ii) For all $x, y \in X$ such that $\partial(x, y) = 2$,*

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} \in \mathrm{Span}(E\hat{x} - E\hat{y}). \tag{3.10}$$

*(iii) There exist $x, y \in X$ such that $\partial(x, y) = 2$ and*

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} \in \mathrm{Span}(E\hat{x} - E\hat{y}). \tag{3.11}$$

*Here $E = E_1$.*

*Proof.* Observe from the proof of Lemma 3.1 that equality is attained in (3.1) if and only if equality is attained in (3.4). We claim $v \neq 0$. This will follow from (3.5) provided we can show $\sigma_0 \neq \sigma_2$. Suppose $\sigma_0 = \sigma_2$. Setting $\theta = \theta_1$ and $\sigma_2 = \sigma_0$ in (2.10) and simplifying the result we find $\theta_1 = -b_1 - 1$. This is inconsistent with (2.14) and $\theta_1 > \theta_d$. We have now shown $\sigma_0 \neq \sigma_2$ and it follows $v \neq 0$. We now see that equality is attained in (3.4) if and only if $u \in \mathrm{Span}(v)$. The result follows. □

# 4 The case of equality

In this section we consider the case of equality in (3.1).

**Lemma 4.1.** *Let $\Gamma$ denote a near polygon distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let $\theta_1$ denote the second largest eigenvalue of $\Gamma$ and let $\sigma_0, \sigma_1, \ldots, \sigma_d$ denote the corresponding cosine sequence. Suppose equality holds in (3.1). Then $\sigma_{i-1} - q\sigma_i$ is independent of $i$ $(1 \leq i \leq d)$, where $q = c_2 - 1$.*

*Proof.* Setting $c_2 = q + 1$ in (3.1) and using $k - a_1 - 1 = b_1$ we find $\theta_1 + 1 = b_1 q^{-1}$. In particular $\theta_1 \neq -1$. Observe $\sigma_1 \neq \sigma_2$; otherwise $\sigma_0 = \sigma_1$ by (2.10) forcing $\theta_1 = k$ by (2.8), a contradiction. Evaluating (2.10) using $\theta_1 + 1 = b_1 q^{-1}$ we find

$$\frac{\sigma_0 - \sigma_1}{\sigma_1 - \sigma_2} = q. \tag{4.1}$$

Fix two vertices $x, y \in X$ with $\partial(x, y) = 2$. Abbreviate $E = E_1$. By Lemma 3.3 there exists $\lambda \in \mathbb{R}$ such that

$$\sum_{z \in A(x,y)} E\hat{z} - \sum_{w \in A(y,x)} E\hat{w} = \lambda(E\hat{x} - E\hat{y}). \tag{4.2}$$

Fix an integer $i$ $(1 \le i \le d-1)$ and pick $u \in X$ with $\partial(u,x) = i-1$ and $\partial(u,y) = i+1$. Taking the inner product of $E\hat{u}$ with both sides of (4.2) and using the fact that $\Gamma$ is a near polygon, we find

$$a_2(\sigma_i - \sigma_{i+1}) = \lambda(\sigma_{i-1} - \sigma_{i+1}). \tag{4.3}$$

Setting $i = 1$ in (4.3) we find $a_2(\sigma_1 - \sigma_2) = \lambda(\sigma_0 - \sigma_2)$. From (4.1) we find $\sigma_0 - \sigma_2 = (\sigma_1 - \sigma_2)(1+q)$. By these comments $\lambda = a_2/(q+1)$. Evaluating (4.3) using this we find

$$\sigma_{i-1} - q\sigma_i = \sigma_i - q\sigma_{i+1} \quad (1 \le i \le d-1).$$

From this we find $\sigma_{i-1} - q\sigma_i$ is independent of $i$ for $1 \le i \le d$.

□

**Lemma 4.2.** *Let $\Gamma$ denote a near polygon distance-regular graph with $d \ge 3$ and intersection numbers $a_1 > 0$, $c_2 > 1$. Let $\theta_1$ denote the second largest eigenvalue of $\Gamma$ and assume equality holds in (3.1). Then $\Gamma$ has classical parameters $(d, q, 0, \beta)$.*

*Proof.* Let the scalar $q$ be as in Lemma 4.1. By Lemma 4.1 we have Theorem 2.4(i) and hence Theorem 2.4(ii). Applying Theorem 2.4(ii) with $i = 2, 3$ we find

$$qc_2 - b_2 - q(qc_1 - b_1) = qc_3 - b_3 - q(qc_2 - b_2). \tag{4.4}$$

Simplifying (4.4) using (2.5) and $c_2 = q+1$, $a_2 = a_1c_2$ we obtain

$$(a_1 + 1 + q)(1 + q + q^2 - c_3) = a_3 - a_1c_3. \tag{4.5}$$

By (2.12) we have $a_3 = a_1c_3$ if $d > 3$, and by (2.13) we have $a_3 \ge a_1c_3$ if $d = 3$. In any case $a_3 \ge a_1c_3$ so the right-hand side of (4.5) is nonnegative. Also $a_1 + 1 + q > 0$ since $q = c_2 - 1$. Evaluating (4.5) using these comments we find

$$c_3 \le 1 + q + q^2. \tag{4.6}$$

By (2.18) and using $c_2 = 1 + q$ we find $c_3 \ge 1 + q + q^2$. Now $c_3 = 1 + q + q^2$ and so $c_3 = (c_2 - q)(1 + q + q^2)$. Applying Theorem 2.5 we find there exist real numbers $\alpha, \beta$ such that $\Gamma$ has classical parameters $(d, q, \alpha, \beta)$. By (2.15) we find $c_2 = (1+q)(1+\alpha)$. By the construction $c_2 = q+1$. Comparing these equations we find $\alpha = 0$. □

**Proof of Theorem 1.1.** The inequality (1.1) is from (3.1).
(i)$\Longrightarrow$(iii). By Lemma 4.2, $\Gamma$ has classical parameters $(d, q, 0, \beta)$. By this and Theorem 2.6 we find $\Gamma$ is a dual polar graph or a Hamming graph.
(iii)$\Longrightarrow$(ii) This is immediate from [2, Corollary 8.5.3].
(ii)$\Longrightarrow$(i) Lemma 3.3(ii) holds by [9, Theorem 3.3], so Lemma 3.3(i) holds and the result follows.

# References


[1] E. Bannai and T. Ito. *Algebraic Combinatorics I: Association Schemes.* Benjamin/Cummings, London, 1984.

[2] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs.* Springer-Verlag, Berlin, 1989.

[3] P. Cameron. Dual polar spaces. *Geom. Dedicata*, 12(1982), 75–85.

[4] Y. Egawa. Characterization of $H(n, q)$ by the parameters. *J. Combin. Theory Ser. A*, 31(1981), 108–125.

[5] A. Neumaier. Characterization of a class of distance-regular graphs. *J. Reine Angew. Math.*, 357(1985), 182–192.

[6] N. Sloane. *An introduction to association schemes and coding theory; in Theory and Application of Special functions (R. Askey, Ed.).* Academic Press, New York, 1975.

[7] D. Stanton. Some $q$-Krawtchouk polynomials on Chevalley groups. *Amer. J. Math.*, 102(4)(1980), 625–662.

[8] P. Terwilliger. Root systems and the Johnson and Hamming graphs. *European J. Combin.*, 8(1987), 73–102.

[9] P. Terwilliger. A new inequality for distance-regular graphs. *Discrete Math.*, 137(1995), 319–332.

[10] C. Weng. Kite-Free $P$- and $Q$-Polynomial Schemes. *Graphs and Combinatorics*, 11(1995), 201-207.

[11] C. Weng. $D$-bounded Distance-regular Graphs. *Europ. J. Combinatorics*, 18(1997), 211–229.

Paul Terwilliger
Department of Mathematics
University of Wisconsin-Madison
Van Vleck Hall
480 Lincoln Drive
Madison, WI 53706–1388
USA
email: `terwilli@math.wisc.edu`

Chih-wen Weng
Department of Applied Mathematics
National Chiao Tung University
1001 Ta Hsueh Road,
Hsinchu
Taiwan 30050
email: `weng@math.nctu.edu.tw`