A Survey on D-bounded Distance-regular Graphs*

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1 Constructions of Weak-geodetically Closed Subgraphs

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$ and distance function ∂. Recall that a sequence x, y, z of vertices of Γ is *geodetic* whenever

$$\partial(x, y) + \partial(y, z) = \partial(x, z).$$

A sequence x, y, z of vertices of Γ is *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1.$$

Definition 1.1. Fix a vertex $x \in X$ and a subset Ω containing x. Ω is *weak-geodetically closed with respect to* x if for any weak-geodetic sequence x, y, z of Γ,

$$z \in \Omega \Longrightarrow y \in \Omega.$$

Ω is *weak-geodetically closed* whenever Ω is weak-geodetically closed with respect to w for any $w \in X$.

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Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [10].

Definition 1.2. Γ is said to be $\bar{i}$-*bounded* whenever for all $x, y \in X$ with $\partial(x, y) = i$, there is a regular weak-geodetically closed subgraph of diameter i which contains x, y. Γ is said to be t-*bounded* whenever Γ is $\bar{i}$-bounded for $0 \leq i \leq t$.

By a *parallelogram of length* i, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x, y) = \partial(z, w) = 1$, $\partial(x, w) = i$, and $\partial(y, w) = \partial(y, z) = \partial(x, z) = i - 1$.

Lemma 1.3. *(Weng[13]) Suppose* Γ *is* t*-bounded. Then* Γ *contains no parallelogram of length* $\leq t + 1$.

Theorem 1.4. *The following (i)-(xi) hold.*

(i) (Weng[13]) Suppose Γ *contains no parallelogram of length* ≤ 3, $c_2 > 1, a_2 \neq 0$. *Then* Γ *is* 2*-bounded.*

(ii) (Weng[13]) Suppose Γ *contains no parallelogram of length* $\leq t + 1$, $c_2 > 1, a_1 \neq 0$. *Then* Γ *is* t*-bounded.*

(iii) (H. Suzuki[11]) Suppose Γ *contains no parallelogram of length* ≤ 3, $c_2 = 1, a_2 > a_1$. *Then* Γ *is* 2*-bounded.*

(iv) (H. Suzuki[11]) Suppose Γ *contains no parallelogram of length* $\leq t+1$, $c_2 = 1, a_2 > a_1 \neq 0$. *Then* Γ *is* t*-bounded.*

(v) (A. Hiraki[3]) Suppose $c_{t-1} = 1, a_{t-1} = a_{t-2} = \cdots = a_1 = 0$, $a_t \neq a_{t-1}$, $c_{2t-1} = 1$. *Then* Γ *is* $\bar{t}$*-bounded.*

(vi) (A. Hiraki[4]) Suppose $c_{t-1} = 1, a_{t-1} = a_{t-2} = \cdots = a_1 \neq 0$, $a_t \neq a_{t-1}$, $c_{2t-1} = 1$. *Then* Γ *is* $\bar{t}$*-bounded.*

(vii) (A. Hiraki[5]) Suppose $c_{t-1} = 1, a_{t-1} = a_{t-2} = \cdots = a_1 \neq 0$, $c_t > 1$ *or* $a_t \neq a_{t-1}$, $a_i = a_1 c_i$ *for* $0 \leq i \leq 2t - 2$. *Then* Γ *is* $\bar{t}$*-bounded.*

(viii) (A. Hiraki[6]) Suppose Γ *contains no parallelogram of length* 2, $a_{t-1} = a_{t-2} = \cdots = a_1$, $c_{t-1} = 1$, $c_t > 1$ *or* $a_t \neq a_{t-1}$. *Then* Γ *is* $\bar{t}$*-bounded.*

(ix) *(A. Hiraki[7]) Suppose* $a_r = a_{r-1} = \cdots = a_1$, $c_{t+r} > 1$, $b_t \neq b_{t-1}$. *Then* Γ *is* $\bar{t}$*-bounded.*

(x) *(A. Hiraki[8]) Suppose* Γ *is a near polygon,* $c_t > c_{t-1} = 1, a_1 \neq 0$. *Then* Γ *is* $\bar{i}$*-bounded for* $t \leq i \leq D - t + 1$.

(xi) *(Y. Pan and Weng[9]) Suppose* Γ *has classical parameters and* $a_2 > a_1 = 0$. *Then* Γ *is* 3*-bounded.*

For two vertices $x, y \in X$ at distance $\partial(x, y) = t$, if there exists a regular weak-geodetically closed subgraph $\Delta(x, y)$ of diameter t containing x, y, then $\Delta(x, y)$ is distance-regular [13]. This implies that

$$\Delta(x, y) = [x, C] := \{w \in X \mid \partial(x, w) + \partial(w, z) = t, \text{ for some } z \in C\},$$

where

$$C \subseteq \{z \mid z \in \Gamma_t(x), \Gamma_1(x) \cap \Gamma_{t+1}(z) = \Gamma_1(x) \cap \Gamma_{t+1}(y)\}$$

is an uncertain union of connected components in $\Gamma_t(x)$. To prove Γ is $\bar{t}$-bounded, it suffices to prove the above $[x, C]$ is regular and weak-geodetically closed with respect to x [13]. In many interesting situations, it happens that C is a single connected component in $\Gamma_t(x)$ containing y. The statement (ii) in the above theorem was proved by making use of this special property. The following problem related to the statement (xi) in the above theorem is still open.

Problem 1.5. Suppose Γ contains no parallelogram of length $\leq t + 1$, and $a_2 > a_1 = 0$. Show Γ is t-bounded.

To show a graph to be $\bar{t}$-bounded property, there are still many open problems related to (v)-(x) in the above theorem. The so called circuit chasing technique is used in this part and some suitable assumptions must be provided to ensure it works.

2 The Poset Constructed from a D-bounded Distance-regular Graph

Let $\Gamma = (X, R)$ be a D-bounded distance-regular graph, where $D \geq 3$ is the diameter of Γ. Let P denote the poset of the elements which are the weak-geodetically closed subgraphs of Γ with partial order by reversed inclusion.

It was shown that P is a ranked atomic meet semi-lattice with rank D. Moreover for two elements $\Delta, \Delta' \in P$, the meet $\Delta \wedge \Delta'$ is the intersection of all weak-geodetically closed subgraphs that contain Δ and Δ', and the join $\Delta \vee \Delta' = \Delta \cap \Delta'$ exists if $\Delta \cap \Delta' \neq \emptyset$. Let P_i denote the rank i elements in P. Then $P_D = X$ and $xy \in R$ whenever the meet $x \wedge y \in P_{D-1}$.

Problem 2.1. Fix an integer i and let G be a graph with vertex set P_i and $\Delta, \Delta' \in P_i$ are adjacent whenever $\Delta \wedge \Delta' \in P_{i-1}$. What is G?

We expect P is some kind of geometry that generalizes the polar spaces. We list some counting results in the study of P.

Theorem 2.2. *The following (i)-(iv) hold.*

(i) (Weng [14])
For $\Delta \in P_{i-1}$ and $\Delta' \in P_{i+1}$ with $\Delta \leq \Delta'$,

$$|[\Delta \cap \Delta'] \cap P_i| = \frac{b_{D-i-1} - b_{D-i+1}}{b_{D-i-1} - b_{D-i}}.$$

(ii) (M. Tsai [12])

$$|P_i| = \frac{b_0 b_1 \cdots b_{D-i-1} (\sum_{t=0}^{D} b_0 b_1 \cdots b_{t-1} c_{t+1} c_{t+2} \cdots c_D)}{b'_0 b'_1 \cdots b'_{D-i-1} (\sum_{t=0}^{D-i} b'_0 b'_1 \cdots b'_{t-1} c_{t+1} c_{t+2} \cdots c_D)},$$

where $b'_t = b_t - b_{D-i}$.

(iii) (Suogang Gao, Jun Guo and Wen Lin [1])
For $\Delta \in P_{i-s}$ and $\Delta' \in P_{i+u}$ with $\Delta \leq \Delta'$,

$$|[\Delta \cap \Delta'] \cap P_i| = \frac{b''_{D-i-u} b''_{D-i-u+1} \cdots b''_{D-i-1}}{b'_{D-i-u} b'_{D-i-u+1} \cdots b'_{D-i-1}},$$

where $b'_t = b_t - b_{D-i}$ and $b''_t = b_t - b_{D-i+s}$.

(iv) (Suogang Gao, Jun Guo, Baohuan Zang, and Lihui Fu [2])
For $\Delta \in P_j$, $\Delta' \in P_i$, $x \in P_D$ such that $\Delta \leq \Delta' \leq x$, the set

$$\{\Delta'' \in P_{i-s+u} \cap [\Delta, x] \mid \Delta' \wedge \Delta'' \in P_{i-s}, \Delta' \vee \Delta'' \in P_{i+u}\}$$

has cardinality

$$\frac{b'_0 b'_1 \cdots b'_{D-i-u-1} b^*_{D-i} b^*_{D-i+1} \cdots b^*_{D-i+s-1}}{b'''_0 b'''_1 \cdots b'''_{D-i-u-1} b^\sharp_{D-i-u} b^\sharp_{D-i-u+1} \cdots b^\sharp_{D-i-u+s-1}},$$

where $b'_t = b_t - b_{D-i}$, $b^*_t = b_t - b_{D-j}$, $b'''_t = b_t - b_{D-i-u}$ *and* $b^\sharp_t = b_t - b_{D-i+s-u}$.

Theorem 2.2(i) is used in the classification of classical distance-regular graphs of negative[14, 15]. Theorem 2.2(ii) has application to construct pooling designs [12]. Theorem 2.2(iv) has an application to the study of Cartesian authentication codes[2].

Problem 2.3. For $i + j - t \leq D$ and $\Delta \in P_i$, compute cardinality of the set

$$\{\Delta' \in P_j \mid \Delta' \wedge \Delta \in P_t, \Delta' \vee \Delta \in P_{i+j-t}\}.$$

3 Further Study

Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$ and distance function ∂. By a *singular line* we mean the intersection of all maximal cliques that contain a given edge.

Definition 3.1. A subset Ω is Δ-*geodetically* closed if Ω is geodetically closed and for any singular line ℓ,

$$|\ell \cap \Omega| \geq 2 \Longrightarrow \ell \subseteq \Omega.$$

Problem 3.2. Suppose any singular line ℓ has the same size, and for a singular ℓ and for any vertex $w \in X$ either there exists a unique vertex in ℓ which is closest to w, or all vertices in ℓ have the same distance to w. Show for any two vertices x, y at distance $\partial(x, y) = t$, there exists a regular Δ-geodetically closed subgraph $\Delta(x, y)$ containing x, y of diameter t.

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