On the Templates Corresponding to Cycle-Symmetric Connectivity in Cellular Neural Networks

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Abstract

In the architecture of cellular neural networks (CNN), connections among cells are built on linear coupling laws. These laws are characterized by the so-called templates which express the local interaction weights among cells. Recently, the complete stability for CNN has been extended from symmetric connections to cycle-symmetric connections. In this presentation, we investigate a class of templates which are obtained from two-dimensional models and have uniform local feedback behaviors. We find necessary and sufficient conditions for the class of templates to have cycle-symmetric connections. The complete stability for CNN is thus concluded.

1 Introduction

We study the cellular neural network (CNN) proposed by Chua and Yang [1988a]. If the model on a two-dimensional $n_1 \times n_2$ array $T_{\mathbf{n}} := \{(i,j) \in \mathbf{Z}^2 \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2\}$ is considered, the circuit equation of a cell is given by

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + \sum_{(k,\ell)\in N_r(i,j)} A(i,j;k,\ell) f_{k,\ell}(x_{k,\ell}) + b_{i,j}, \qquad (i,j) \in T_{\mathbf{n}}, \qquad (1.1)$$

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where $N_r(i,j)$ represents the r-neighborhood of the cell at (i,j). That is

$$N_r(i,j) = \{(k,\ell)|(k,\ell) \in T_{\mathbf{n}}, \text{ and } \max(|k-i|, |\ell - j|) = r\}.$$

The feedback operator is represented by real numbers $A(i,j;k,\ell)$, $(i,j) \in T_{\mathbf{n}}$, $(k,\ell) \in N_r(i,j)$, and these real numbers constitute the template for CNN. This template describes the connection weights among cells. For

$$(i,j) \in \{(1,s), (\tau,1), (n_1,s), (\tau,n_2) \ \ | \ 1 \le s \le n_2, 1 \le \tau \le n_1\},$$

the $x_{i,j}$ term in (1.1) has to be determined from the imposed boundary condition. In addition, each $f_{i,j}$ is called an output function. The standard output function is given by $f_{i,j}(\xi) = f(\xi) := \frac{1}{2}(|\xi+1| + |\xi - 1|)$. $b_{i,j}$ represents the terms from the control operator and threshold. For details, please see [Chua & Yang, 1988], [Chua, 1998].

Equation (1.1) can be written in the following matrix form.

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= -\mathbf{x} + \mathbf{A}\mathbf{y} + \mathbf{b} \\ \mathbf{y} &:= \mathbf{F}(\mathbf{x}), \end{aligned} \tag{1.2}$$

where $\mathbf{A}$ is an $n \times n$ matrix ($n = n_1 \times n_2$), $\mathbf{x}$ is a vector variable, $\mathbf{b}$ is a constant vector, $\mathbf{F}$ is a vector function with vector variable domain, and the index set of these matrices is $T_{\mathbf{n}}$. Note that the $((i,j),(k,\ell))$ entry of $\mathbf{A}$ is

$$\begin{cases} A(i,j;k,\ell) & \text{if } (k,\ell) \in N_r(i,j), \\ 0 & \text{otherwise.} \end{cases} \tag{1.3}$$

For notation convenience, if $u = (i,j), v = (k,\ell)$, we write $A(u;v)$ or $A(i,j;k,\ell)$ for the (u,v) entry of $\mathbf{A}$. $\mathbf{A}$ is called the *template* for CNN.

The complete stability for (1.1) and (1.2) with standard output function has been studied in [Chua and Yang, 1988], [Lin and Shih, 1999]. The basic assumption is the symmetry condition of the circuit parameters in (1.2):

$$A(i,j;k,\ell) = A(k,\ell;i,j), \quad \text{for all } (k,\ell),(i,j) \in T_{\mathbf{n}}. \tag{1.4}$$

With this assumption, if (1.1) is imposed with certain discrete-type boundary conditions, $\mathbf{A}$ is always symmetric, as (1.2) is reformulated into the form (1.1). Recently, the complete stability for (1.2) (hence (1.1) has been extended to more general sigmoidal output functions and the so-called cycle-symmetric connections, *cf.* [Shih, 2001]. Cycle-symmetric matrices can be described as follows. Let $\mathbf{A} = [\mathbf{A}(u;v)]$ be an $n \times n$ matrix with either $\mathbf{A}(u;v) = 0$ or $\mathbf{A}(u;v)\mathbf{A}(v;u) \neq 0$ for $u, v \in T_{\mathbf{n}}$. There corresponds an undirected graph whose vertex v is joined to the vertex u by the edge uv if and only if $\mathbf{A}(u;v) \neq 0$ and $\mathbf{A}(v;u) \neq 0$. With the abuse of notation, we denote this graph also by $T_{\mathbf{n}}$. Let $u_1, \cdots, u_n$ be n distinct vertices. Then the sequence $u_1 u_2 \cdots u_\ell u_1$ is a *cycle* if any two consecutive vertices have an edge. Sometimes we treat the cycle as the edge set $\{u_1u_2, u_2u_3, \cdots, u_{\ell-1}u_\ell, u_\ell u_1$. $\mathbf{A}$ is *cycle-symmetric* if $\mathbf{A}$ satisfies the following two conditions $(H_1) - (H_2)$.

$$(H_1) \qquad \mathbf{A}(u;v)\mathbf{A}(v;u) > 0, \text{ if } \mathbf{A}(u;v) \neq 0, \tag{1.5}$$

$$(H_2) \quad \prod_{uv \in C} \mathbf{A}(u;v) = \prod_{uv \in C} \mathbf{A}(v;u), \text{ for any cycle } C, \tag{1.6}$$

where Π denotes the product. $\mathbf{A}$ is called *sign-symmetric* if $\mathbf{A}$ satisfies (H_1). It is straightforward to verify that symmetric $\mathbf{A}$ satisfies (H_1) and (H_2).

2 Preliminaries

In this presentation, we plan to investigate the templates which have uniform local behaviors. More precisely, we assume $\mathbf{A}$ satisfying the following special form. Fix a 3×3 matrix

$$\mathbf{M} = \begin{pmatrix} m(-1;1) & m(0;1) & m(1;1) \\ m(-1;0) & m(0;0) & m(1;0) \\ m(-1;-1) & m(0;-1) & m(1;-1) \end{pmatrix}. \tag{2.7}$$

Suppose

$$A(i,j;k,\ell) = \begin{cases} m(k-i;\ell-j) & \text{if } |k-i| \leq 1 \text{ and } |\ell-j| \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Observe that this is the case $r = 1$ in equation (1.1), and (1.1) can be rewritten as

$$\frac{dx_{i,j}}{dt} = -x_{i,j} + \sum_{(k,\ell) \in N_1(i,j)} m(k-i;\ell-j) f_{k,\ell}(x_{k,\ell}) + b_{i,j}, \qquad (i,j) \in T_{\mathbf{n}}.$$

Since $\mathbf{A}$ is determined by $\mathbf{M}$, we also called $\mathbf{M}$ the template for CNN. Note that if $m(i;j) = m(-i;-j)$ then $\mathbf{A}$ is a symmetric matrix. Recall that

$$A(u;v) \neq 0 \text{ iff } uv \text{ is an edge in the graph } T_{\mathbf{n}}. \tag{2.9}$$

In this case, $m(v-u) \neq 0$ by (2.8), and we call the edge uv has *type* $v-u$, where the subtraction is the usual vector subtraction. Type $u-v$ is called the *opposite* type of $v-u$.

We shall determine all the templates $\mathbf{A}$ in (2.8) to be cycle-symmetric in next section. The complete stability for CNN in these templates is thus concluded.

3 The Main Result

Throughout this section, we suppose the template $\mathbf{A}$ satisfies (2.8), and suppose

$$m(i;j) = 0 \text{ if and only if } m(-i;-j) = 0 \quad (-1 \le i, j \le 1). \tag{3.10}$$

The assumption (3.10) on M ensures the condition $A(i,j;k,\ell) = 0$ if and only if $A(k,\ell;i,j) = 0$. The following lemma is immediate from (1.5), (2.8) and (3.10).

Lemma 3.1. $\mathbf{A}$ is sign-symmetric if and only if $m(i;j)m(-i;-j) \ge 0 \quad (-1 \le i, j \le 1)$.

Suppose $\mathbf{A}$ is sign-symmetric. We shall prove that the following conditions (i)-(iv) are equivalent to $\mathbf{A}$ being cycle symmetric.

Conditions:

(i) $m(0;1)m(1;0)m(-1;-1) = m(1;1)m(-1;0)m(0;-1)$;

(ii) $m(1;0)m(0;-1)m(-1;1) = m(1;-1)m(0;1)m(-1;0)$;

(iii) $m(0;-1)^2 m(-1;1)m(1;1) = m(-1;-1)m(1;-1)m(0;1)^2$;

(iv) $m(-1;0)^2 m(1;-1)m(1;1) = m(-1;-1)m(-1;1)m(1;0)^2$.

Notably, in (i), $m(0;1)m(1;0)m(-1;-1) = m(1;1)m(-1;0)m(0;-1)$ is exactly the condition that $\mathbf{M}$ is cycle-symmetric, and in (ii), $m(1;0)m(0;-1)m(-1;1) = m(1;-1)m(0;1)m(-1;0)$ is the condition that

$$\mathbf{M}^t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

is cycle-symmetric, where $\mathbf{M}^t$ is the transpose of M. See (1.6) and (2.7). We need the following technical lemma to prove our main result. This lemma says that a cycle C of $T_{\mathbf{n}}$ of length at least 5 can be cut into two cycles by a path of $T_{\mathbf{n}}$. We omit the proof which is not particularly illuminating.

Lemma 3.2. Let $u_1 u_2 \cdots u_s$ be a cycle C of length $s \ge 5$ in $T_{\mathbf{n}}$. Then there is a path $P: v_1 \cdots v_\tau$ in $T_{\mathbf{n}}$ such that the following (a)-(b) hold:

(a) $v_1 = u_p$ and $v_\tau = u_q$ for some p, q $(1 \le p < q - 1 \le s - 1)$.

(b) If $\tau = 2$, then $v_1 v_2 \neq u_1 u_s$. If $\tau > 2$, then $v_2, \cdots, v_{\tau-1}$ are inside the open bounded region of C.

In fact, the path P in Lemma 3.2 can be chosen such that each of its edge has the same or the opposite type of some edge in the cycle C.

Theorem 3.3. Suppose $\mathbf{A}$ is sign-symmetric. Then $\mathbf{A}$ is cycle-symmetric if and only if Conditions (i)-(iv) hold.

Proof. ($\Longrightarrow$) Suppose $\mathbf{A}$ is cycle-symmetric. To prove (i), set $u_1 = (1,1)$, $u_2 = (1,2), u_3 = (2,2)$, and apply (2.8) to (1.6) with the cycle $u_1u_2u_3u_1$. Similarly, set $u_1 = (1,2)$, $u_2 = (2,2)$, $u_3 = (2,1)$, and consider the cycle $u_1u_2u_3u_1$ to prove (ii). To prove (iii), set $u_1 = (2,3)$, $u_2 = (2,2)$, $u_3 = (2,1)$, $u_4 = (1,2)$, and apply (2.8) to (1.6) with the cycle $u_1u_2u_3u_4u_1$. Similarly, to prove (iv), set $u_1 = (3,2)$, $u_2 = (2,2)$, $u_3 = (1,2)$, $u_4 = (2,1)$, and consider the cycle $u_1u_2u_3u_4u_1$.

($\Longleftarrow$) Suppose (i)-(iv) hold. We shall prove (1.6) for any cycle $C: \; u_1 \cdots u_s u_1$ of $T_{\mathbf{n}}$. Note that the area of enclosed region of C is $V = \frac{k}{2}$ for some positive integer k. We prove by induction on k. Suppose $k = 1$. observe this is equivalent to $s = 3$, and $u_1u_2u_3u_1$ is a triangle. There are at most 8 possible triangles of area $\frac{1}{2}$ starting with u_1. For each triangle, we find Conditions (i)-(ii) are enough to obtain (1.6) with the cycle C. To prove the case $k = 2$, we prove a more general case that $s = 4$. There are two essential three types of cycles, a square of area 1, a triangle of area 1 with the base of length 2, or a square of area 2. For squares, (1.6) is a trivial equality. For a triangle , Conditions (iii)-(iv) are enough to obtain (1.6) with the cycle C. Now suppose $k > 2$ and $s \ge 5$. Let $P = v_1 \cdots v_\tau$ be a path satisfying (a)-(b) in Lemma 3.2. Define the following 3 paths P_1, P_2, P_3.

$$\begin{array}{lll} P_1 & : & u_1 u_2 \cdots u_p \\ P_2 & : & u_p u_{p+1} \cdots u_q \\ P_3 & : & u_q u_{q+1} \cdots u_s u_1 \end{array} \tag{3.11}$$

Let P^{-1} be the reversed path of P. Note that $P_1 \cup P \cup P_3$ and $P^{-1} \cup P_2$ both are cycles with smaller enclosed areas. By induction,

$$\prod_{u_j u_{j+1} \in P_1 \cup P \cup P_3} A(u_j; u_{j+1}) = \prod_{u_j u_{j+1} \in P_1 \cup P \cup P_3} A(u_{j+1}; u_j), \tag{3.12}$$

and

$$\prod_{u_j u_{j+1} \in P^{-1} \cup P_2} A(u_j; u_{j+1}) = \prod_{u_j u_{j+1} \in P^{-1} \cup P_2} A(u_{j+1}; u_j). \tag{3.13}$$

Multiplying the same sides of (3.12)-(3.13) together, we have

$$\prod_{u_j u_{j+1} \in P_1 \cup P_2 \cup P_3 \cup P \cup P^{-1}} A(u_j; u_{j+1}) = \prod_{u_j u_{j+1} \in P_1 \cup P_2 \cup P_3 \cup P \cup P^{-1}} A(u_{j+1}; u_j). \tag{3.14}$$

Deleting the common term

$$\prod_{u_j u_{j+1} \in P \cup P^{-1}} A(u_j; u_{j+1}) = \prod_{u_j u_{j+1} \in P \cup P^{-1}} A(u_{j+1}; u_j). \tag{3.15}$$

which is nonzero ensured by (2.9), we have (1.6).

Corollary 3.3. Suppose A is sign-symmetric, and $m(0;1)m(-1;0) \neq 0$. Then $\mathbf{A}$ is cycle-symmetric if and only if Conditions (i)-(ii) hold.

Proof. Multiply the same sides of the equalities in Conditions (i)-(ii), we obtain

$$\begin{array}{rl} & m(1;0)^2 m(-1;-1)m(-1;1)m(0;1)m(0;-1) \\ = & m(-1;0)^2 m(1;1)m(1;-1)m(0;1)m(0;-1). \end{array} \tag{3.16}$$

Observe $m(0;1)m(0;-1) \neq 0$ by Lemma 3.1. Dividing both sides of (3.16) by $m(0;1)m(0;-1)$, we have (iv). Multiply the opposite sides of equalities in Conditions (i)-(ii), we obtain

$$\begin{array}{rl} & m(0;1)^2 m(-1;-1)m(1;-1)m(-1;0)m(1;0) \\ = & m(0;-1)^2 m(-1;1)m(1;1)m(-1;0)m(1;0). \end{array} \tag{3.17}$$

Observe $m(-1;0)m(1;0) \neq 0$. Dividing both sides of (3.17) by $m(0;-1)m(0;1)$, we obtain (iii).

Combining Theorem 3.3 with Shih[2001], we have the following conclusion.

Corollary 2.4. The complete stability of CNN holds for any sign-symmetric templates which satisfy Conditions (i)-(iv). In particular, the complete stability of CNN holds for the following sign-symmetric square-cross and diagonal-cross templates:

$$\mathbf{M} = \begin{pmatrix} 0 & m(0;1) & 0 \\ m(-1;0) & m(0;0) & m(1;0) \\ 0 & m(0;-1) & 0 \end{pmatrix}, \ \mathbf{M} = \begin{pmatrix} m(-1;1) & 0 & m(1;1) \\ 0 & m(0;0) & 0 \\ m(-1;-1) & 0 & m(1;-1) \end{pmatrix},$$

where $m(i;j)m(-i;-j) \geq 0$.

Remarks.

(a) For the case of symmetric template, that is, $m(i;j) = m(-i;-j)$ $(-1 \leq i, j \leq 1)$ (i)-(iv) hold clearly. Thus, cycle-symmetric connectivity indeed generalizes symmetric connectivity.

(b)

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & m(0;0) & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

satisfies Conditions (i)-(iii), but does not satisfies Condition (iv). This shows (iv) is necessary in Theorem 2.2. Similarly, (iii) is necessary.

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