Triangle-free Distance-regular Graphs *

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April 28, 2006

Abstract

Let Γ denote a distance-regular graph with $d \geq 3$. By a *parallelogram of length* 3, we mean a 4-tuple $xyzw$ consisting of vertices in Γ such that $\partial(x,y) = \partial(z,w) = 1$, $\partial(x,z) = 3$, and $\partial(x,w) = \partial(y,w) = \partial(y,z) = 2$, where ∂ denotes the path-length distance function. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. We prove the following (i), (ii) are equivalent. (i) Γ is Q-polynomial and contains no parallelograms of length 3; (ii) Γ has classical parameters (d, b, α, β). Furthermore, suppose (i), (ii) hold we show that each of $b(b+1)^2(b+2)/c_2$, $(b-2)(b-1)b(b+1)/(2+2b-c_2)$ is an integer and that $c_2 \leq b(b+1)$.

Keywords: Distance-regular graph, Q-polynomial, classical parameters.

1 Introduction

Let Γ denote a distance-regular graph with diameter $d \geq 3$ (See Section 2 for formal definitions.). It is known that if Γ has classical parameters then Γ is Q-polynomial [2, Corollary 8.4.2]. The converse is not true, since an ordinary n-gon has the Q-polynomial property, but without classical parameters

*Work partially supported by the National Science Council of Taiwan, R.O.C.

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[2, Table 6.6]. Many authors prove the converse under various additional assumptions. Indeed assume Γ is Q-polynomial. Then Brouwer, Cohen, Neumaier in [2, Theorem 8.5.1] show that if Γ is a near polygon, with the intersection number $a_1 \neq 0$, then Γ has classical parameters. Weng generalizes this result with a weaker assumption, without kites of length 2 or length 3 in Γ, to replace the near polygon assumption [7, Lemma 2.4]. For the complement case $a_1 = 0$, Weng shows that Γ has classical parameters if (i) Γ contains no parallelograms of length 3 and no parallelograms of length 4; (ii) Γ has the intersection number $a_2 \neq 0$; and (iii) Γ has diameter $d \geq 4$ [8, Theorem 2.11]. Our first theorem improves the above result.

Theorem 1.1. *Let Γ denote a distance-regular graph with diameter $d \geq 3$ and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Then the following (i)-(iii) are equivalent.*

(i) Γ is Q-polynomial and Γ contains no parallelograms of length 3.

(ii) Γ is Q-polynomial and Γ contains no parallelograms of any length i for $3 \leq i \leq d$.

(iii) Γ has classical parameters (d, b, α, β) with $b < -1$.

Many authors study distance-regular graph Γ with $a_1 = 0$ and other additional assumptions. For example, Miklavič assumes Γ is Q-polynomial and shows Γ is 1-homogeneous [11]; Koolen and Moulton assume Γ has degree 8, 9 or 10 and show that there are finitely many such graphs [12]; Jurišić, Koolen and Miklavič assume Γ has an eigenvalue with multiplicity equal to the valency, $a_2 \neq 0$, and the diameter $d \geq 4$ to show $a_4 = 0$ and Γ is 1-homogeneous [13]. In the second theorem, we assume Γ has classical parameters and obtain the following.

Theorem 1.2. *With the notation and assumptions of Theorem 1.1, suppose (i)-(iii) hold. Then each of*

$$\frac{b(b+1)^2(b+2)}{c_2}, \quad \frac{(b-2)(b-1)b(b+1)}{2+2b-c_2} \tag{1.1}$$

is an integer. Moreover

$$c_2 \leq b(b+1). \tag{1.2}$$

2 Preliminaries

In this section we review some definitions and basic concepts concerning distance-regular graphs. See Bannai and Ito [1] or Terwilliger [6] for more background information.

Let Γ=$(X,\, R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter d:=$max\{\, \partial(x,y) \mid x, y \in X\}$.

For a vertex $x \in X$ and $0 \leq i \leq d$, set $\Gamma_i(x) = \{\, y \in X \mid \; \partial(x,y) = i\}$. Γ is said to be *distance-regular* whenever for all integers $0 \leq h, i, j \leq d$, and all vertices $x, y \in X$ with $\partial(x,y) = h$, the number

$$p^h_{ij} = |\{\, z \in X \mid \; z \in \Gamma_i(x) \cap \Gamma_j(y)\}|$$

is independent of x, y. The constants p^h_{ij} are known as the *intersection numbers* of Γ. For convenience, set $c_i := p^i_{1\; i-1}$ for $1 \leq i \leq d$, $a_i := p^i_{1\; i}$ for $0 \leq i \leq d$, $b_i := p^i_{1\; i+1}$ for $0 \leq i \leq d-1$, and put $b_d := 0$, $c_0 := 0$, $k := b_0$. Note that k is called the valency of Γ. It is immediately from the definition that $b_i \neq 0$ for $0 \leq i \leq d-1$ and $c_i \neq 0$ for $1 \leq i \leq d$. Moreover

$$k = a_i + b_i + c_i \quad \text{for} \;\; 0 \leq i \leq d. \tag{2.1}$$

From now on we assume Γ is distance-regular with diameter $d \geq 3$.

Let $\mathbb{R}$ denote the real number field. Let $\mathrm{Mat}_X(\mathbb{R})$ denote the algebra of all the matrices over $\mathbb{R}$ with the rows and columns indexed by the elements of X. For $0 \leq i \leq d$ let A_i denote the matrices in $\mathrm{Mat}_X(\mathbb{R})$, defined by the rule

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i; \\ 0, & \text{if } \partial(x,y) \neq i \end{cases} \qquad \text{for} \;\; x, y \in X.$$

We call A_i the *distance matrices* of Γ. We have

$$A_0 = I, \tag{2.2}$$

$$A_0 + A_1 + \cdots + A_d = J \quad \text{where} \;\; J = \text{all } 1's \text{ matrix}, \tag{2.3}$$

$$A_i^t = A_i \qquad \text{for} \;\; 0 \leq i \leq d, \tag{2.4}$$

$$A_i A_j = \sum_{h=0}^{d} p^h_{ij} A_h \quad \text{for} \;\; 0 \leq i, j \leq d, \tag{2.5}$$

$$A_i A_j = A_j A_i \quad \text{for} \;\; 0 \leq i, j \leq d. \tag{2.6}$$

Let M denote the subspace of $\mathrm{Mat}_X(\mathbb{R})$ spanned by $A_0, A_1, \ldots, A_d$. Then M is a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{R})$, and is known as the *Bose-Mesner algebra* of Γ. By [2, p. 59, 64], M has a second basis $E_0, E_1, \ldots, E_d$ such that

$$E_0 = |X|^{-1} J, \tag{2.7}$$

$$E_i E_j = \delta_{ij} E_i \quad \text{for} \ \ 0 \le i, j \le d, \tag{2.8}$$

$$E_0 + E_1 + \cdots + E_d = I, \tag{2.9}$$

$$E_i^t = E_i \quad \text{for} \ \ 0 \le i \le d. \tag{2.10}$$

The $E_0, E_1, \ldots, E_d$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent. Let E denote any primitive idempotent of Γ. Then we have

$$E = |X|^{-1} \sum_{i=0}^{d} \theta_i^* A_i \tag{2.11}$$

for some $\theta_0^*, \theta_1^*, \ldots, \theta_d^* \in \mathbb{R}$, called the *dual eigenvalues* associated with E.

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of V as being indexed by X. Then the Bose-Mesner algebra M acts on V by left multiplication. We call V the *standard module* of Γ. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \cdots, 0, 1, 0, \cdots, 0)^t, \tag{2.12}$$

where the 1 is in coordinate x. Also, let $\langle \, , \rangle$ denote the dot product

$$\langle u, v \rangle = u^t v \quad \text{for} \ \ u, v \in V. \tag{2.13}$$

Then referring to the primitive idempotent E in (2.11), we compute from (2.10)-(2.13) that for x, $y \in X$,

$$\langle E\hat{x}, \hat{y} \rangle = |X|^{-1} \theta_i^* \tag{2.14}$$

where $i = \partial(x, y)$.

Let $\circ$ denote entry-wise multiplication in $\mathrm{Mat}_X(\mathbb{R})$. Then

$$A_i \circ A_j = \delta_{ij} A_i \quad \text{for} \ \ 0 \le i, j \le d,$$

so M is closed under $\circ$. Thus there exists $q_{ij}^k \in \mathbb{R}$ for $0 \leq i, j, k \leq d$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{k=0}^{d} q_{ij}^k E_k \quad \text{for} \ \ 0 \leq i, j \leq d.$$

Γ is said to be Q-*polynomial* with respect to the given ordering E_0, E_1,. . ., E_d of the primitive idempotents, if for all integers $0 \leq h, i, j \leq d$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any primitive idempotent of Γ. Then Γ is said to be Q-polynomial with respect to E whenever there exists an ordering E_0, $E_1 = E$,. . ., E_d of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial. If Γ is Q-polynomial with respect to E, then the associated dual eigenvalues are distinct [5, p. 384].

The following theorem about the Q-polynomial property will be used in this paper.

Theorem 2.1. *[6, Theorem 3.3] Assume Γ is Q-polynomial with respect to a primitive idempotent E, and let $\theta_0^*, \ldots, \theta_d^*$ denote the corresponding dual eigenvalues. Then the following (i), (ii) hold.*

(i) For all integers $1 \leq h \leq d$, $0 \leq i, j \leq d$ and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$\sum_{\substack{z \in X \\ \partial(x,z)=i \\ \partial(y,z)=j}} E\hat{z} - \sum_{\substack{z \in X \\ \partial(x,z)=j \\ \partial(y,z)=i}} E\hat{z} = p_{ij}^h \frac{\theta_i^* - \theta_j^*}{\theta_0^* - \theta_h^*} (E\hat{x} - E\hat{y}). \tag{2.15}$$

(ii) For an integer $3 \leq i \leq d$,

$$\theta_{i-2}^* - \theta_{i-1}^* = \sigma(\theta_{i-3}^* - \theta_i^*) \tag{2.16}$$

for appropriate $\sigma \in \mathbb{R} \setminus \{0\}$.

Γ is said to have *classical parameters* (d, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \quad \text{for} \ \ 0 \leq i \leq d, \tag{2.17}$$

$$b_i = \left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \quad \text{for} \ \ 0 \leq i \leq d, \tag{2.18}$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}. \tag{2.19}$$

Suppose Γ has classical parameters (d, b, α, β). Combining (2.17)-(2.19), we have

$$\begin{aligned} a_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(\beta - 1 + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \\ &= \begin{bmatrix} i \\ 1 \end{bmatrix}\left(a_1 + \alpha\left(1 - \begin{bmatrix} i \\ 1 \end{bmatrix} - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right)\right) \quad \text{for } \ 0 \le i \le d. \end{aligned} \tag{2.20}$$

Note that if Γ has classical parameters (d, b, α, β) and $d \ge 3$, then b is an integer and $b \ne 0, -1$ [2, Proposition 6.2.1]. Γ is said to have *classical parameters* if Γ has classical parameters (d, b, α, β) for some constants d, b, α, β. It is shown that a distance-regular graph with classical parameters has the Q-polynomial property [2, Theorem 8.4.1]. Terwilliger generalizes this to the following.

Theorem 2.2. *[6, Theorem 4.2] Let Γ denote a distance-regular graph with diameter $d \ge 3$. Choose $b \in \mathbb{R} \setminus \{0, -1\}$. Then the following (i), (ii) are equivalent.*

(i) Γ is Q-polynomial with associated dual eigenvalues $\theta^*_0, \theta^*_1, \ldots, \theta^*_d$ satisfying

$$\theta^*_i - \theta^*_0 = (\theta^*_1 - \theta^*_0)\begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for } \ 1 \le i \le d.$$

(ii) Γ has classical parameters (d, b, α, β) for some real constants α, β.

Pick an integer $2 \le i \le d$. By a *parallelogram* of length i in Γ, we mean a 4-tuple $xyzw$ of vertices of X such that

$$\partial(x, y) = \partial(z, w) = 1, \quad \partial(x, z) = i,$$

$$\partial(x, w) = \partial(y, w) = \partial(y, z) = i - 1.$$

See Figure 1.

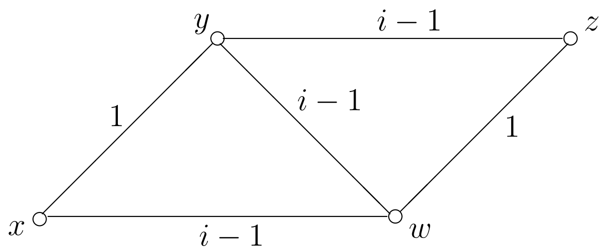

Figure 1: A parallelogram of length i.

3 Proof of Theorem 1.1

In this section we prove our first main theorem. We start with a lemma.

Lemma 3.1. *Let Γ denote a Q-polynomial distance-regular graph with diameter $d \geq 3$ and intersection number $a_1 = 0$. Fix an integer i for $2 \leq i \leq d$ and three vertices x, y, z such that*

$$\partial(x, y) = 1, \quad \partial(y, z) = i - 1, \quad \partial(x, z) = i.$$

Then the quantity

$$s_i(x, y, z) := |\Gamma_{i-1}(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)| \tag{3.1}$$

is equal to

$$a_{i-1} \frac{(\theta^*_0 - \theta^*_{i-1})(\theta^*_2 - \theta^*_i) - (\theta^*_1 - \theta^*_{i-1})(\theta^*_1 - \theta^*_i)}{(\theta^*_0 - \theta^*_{i-1})(\theta^*_{i-1} - \theta^*_i)}. \tag{3.2}$$

In particular (3.1) is independent of the choice of the vertices x, y, z.

Proof. Let $s_i(x, y, z)$ denote the expression in (3.1) and set

$$\ell_i(x, y, z) = |\Gamma_i(x) \cap \Gamma_{i-1}(y) \cap \Gamma_1(z)|.$$

Observe

$$s_i(x, y, z) + \ell_i(x, y, z) = a_{i-1}. \tag{3.3}$$

By (2.15) we have

$$\sum_{\substack{w \in X \\ \partial(y,w)=i-1 \\ \partial(z,w)=1}} E\hat{w} - \sum_{\substack{w \in X \\ \partial(y,w)=1 \\ \partial(z,w)=i-1}} E\hat{w} = a_{i-1} \frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_{i-1}} (E\hat{y} - E\hat{z}). \tag{3.4}$$

Taking the inner product of (3.4) with $\hat{x}$ using (2.14) and the assumption $a_1 = 0$, we obtain

$$s_i(x,y,z)\theta^*_{i-1} + \ell_i(x,y,z)\theta^*_i - a_{i-1}\theta^*_2 = a_{i-1}\frac{\theta^*_{i-1} - \theta^*_1}{\theta^*_0 - \theta^*_{i-1}}(\theta^*_1 - \theta^*_i). \tag{3.5}$$

Solving $s_i(x,y,z)$ by using (3.3) and (3.5), we get (3.2). □

From Lemma 3.1, $s_i(x,y,z)$ is a constant for any vertices x, y, z with $\partial(x,y) = 1$, $\partial(y,z) = i-1$, $\partial(x,z) = i$.

Definition 3.2. We let s_i denote the expression in (3.1). Note that $s_i = 0$ if and only if Γ contains no parallelograms of length i.

Lemma 3.3. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β). Suppose intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $\alpha < 0$ and $b < -1$.*

Proof. From (2.19), (2.20) and since $a_1 = 0$, $a_2 \neq 0$, we have

$$-\alpha(b+1)^2 = a_2 - (b+1)a_1 = a_2 > 0. \tag{3.6}$$

Hence $b \neq -1$ and

$$\alpha < 0. \tag{3.7}$$

By direct calculation from (2.17), we get

$$(c_2 - b)(b^2 + b + 1) = c_3 > 0. \tag{3.8}$$

Since b is an integer and $b \neq 0, -1$ [2, Proposition 6.2.1], we have

$$b^2 + b + 1 > 0. \tag{3.9}$$

Then from (3.8), implies

$$c_2 > b. \tag{3.10}$$

By using (2.17), (3.10), we get

$$\alpha(1+b) = c_2 - b - 1 \geq 0. \tag{3.11}$$

Hence $b < -1$ by (3.7) and since $b \neq -1$. □

Proof of Theorem 1.1:

(ii)⇒(i) This is clear.

(iii)⇒(ii) Suppose Γ has classical parameters. Then Γ is Q-polynomial with associated dual eigenvalues $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ satisfying

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for} \ \ 1 \leq i \leq d. \tag{3.12}$$

We need to prove $s_i = 0$ for $3 \leq i \leq d$. To compute s_i in (3.2), observe from (3.12) that

$$\theta_{i-1}^* - \theta_i^* = (\theta_0^* - \theta_1^*) b^{1-i} \quad \text{for } 1 \leq i \leq d. \tag{3.13}$$

Summing (3.13) for consecutive i, we find

$$(\theta_1^* - \theta_i^*) = (\theta_0^* - \theta_1^*)(b^{-1} + b^{-2} + \cdots + b^{1-i}), \tag{3.14}$$

$$(\theta_1^* - \theta_{i-1}^*) = (\theta_0^* - \theta_1^*)(b^{-1} + b^{-2} + \cdots + b^{2-i}), \tag{3.15}$$

$$(\theta_2^* - \theta_i^*) = (\theta_0^* - \theta_1^*)(b^{-2} + b^{-3} + \cdots + b^{1-i}), \tag{3.16}$$

$$(\theta_0^* - \theta_{i-1}^*) = (\theta_0^* - \theta_1^*)(b^{0} + b^{-1} + \cdots + b^{2-i}) \tag{3.17}$$

for $3 \leq i \leq d$. Evaluating (3.2) by using (3.13)-(3.17), we find $s_i = 0$ for $3 \leq i \leq d$.

(i)⇒(iii) Observe $s_3 = 0$. Then by setting $i = 3$ in (3.2) and using the assumption $a_2 \neq 0$, we find

$$(\theta_0^* - \theta_2^*)(\theta_2^* - \theta_3^*) - (\theta_1^* - \theta_2^*)(\theta_1^* - \theta_3^*) = 0. \tag{3.18}$$

Set

$$b := \frac{\theta_1^* - \theta_0^*}{\theta_2^* - \theta_1^*}. \tag{3.19}$$

Then

$$\theta_2^* = \theta_0^* + \frac{(\theta_1^* - \theta_0^*)(b+1)}{b}. \tag{3.20}$$

Eliminating θ_2^*, θ_3^* in (3.18) using (3.20) and (2.16), we have

$$\frac{-(\theta_1^* - \theta_0^*)^2(\sigma b^2 + \sigma b + \sigma - b)}{\sigma b^2} = 0. \tag{3.21}$$

for appropriate $\sigma \in \mathbb{R} \setminus \{0\}$. Since $\theta_1^* \neq \theta_0^*$,

$$\sigma b^2 + \sigma b + \sigma - b = 0,$$

and hence

$$\sigma^{-1} = \frac{b^2 + b + 1}{b}. \tag{3.22}$$

From Theorem 2.2, to prove that Γ has classical parameter, it suffices to prove that

$$\theta_i^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} i \\ 1 \end{bmatrix} b^{1-i} \quad \text{for} \ \ 1 \leq i \leq d. \tag{3.23}$$

We prove (3.23) by induction on i. The case $i = 1$ are trivial and case $i = 2$ is from (3.20). Now suppose $i \geq 3$. Then (2.16) implies

$$\theta_i^* = \sigma^{-1}(\theta_{i-1}^* - \theta_{i-2}^*) + \theta_{i-3}^* \tag{3.24}$$

Evaluate (3.24) using (3.22) and the induction hypothesis, we find $\theta_i^* - \theta_0^*$ is as in (3.23). Therefore, Γ has classical parameters (d, b, α, β) for some scalars α, β. Note that $b < -1$ from Lemma 3.3. □

4 Proof of Theorem 1.2

Recall that a sequence x, y, z of vertices of Γ are *geodetic* whenever

$$\partial(x, y) + \partial(y, z) \quad = \quad \partial(x, z).$$

Recall that a sequence x, y, z of vertices of Γ are *weak-geodetic* whenever

$$\partial(x, y) + \partial(y, z) \leq \partial(x, z) + 1.$$

Definition 4.1. A subset $\Omega \subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, y, z of Γ,

$$x,\, z \in \Omega \Longrightarrow y \in \Omega.$$

Theorem 4.2. *[9, Proposition 6.7, Theorem 4.6] Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $d \geq 3$. Assume that the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Suppose that Γ contains no parallelograms of length 3. Then for each pair of vertices $v, w \in X$ at distance $\partial(v, w) = 2$, there exists a weak-geodetically closed subgraph Ω of diameter 2 in Γ containing v, w. Furthermore Ω is strongly regular with intersection numbers*

$$a_i(\Omega) = a_i(\Gamma), \tag{4.1}$$
$$c_i(\Omega) = c_i(\Gamma), \tag{4.2}$$
$$b_i(\Omega) = a_2(\Gamma) + c_2(\Gamma) - a_i(\Omega) - c_i(\Omega) \tag{4.3}$$

for $0 \leq i \leq 2$.

Corollary 4.3. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then there exists a weak-geodetically closed subgraph Ω of diameter 2. Furthermore the intersection numbers of Ω satisfy*

$$b_0(\Omega) = (1 + b)(1 - \alpha b), \tag{4.4}$$
$$b_1(\Omega) = b(1 - \alpha - \alpha b), \tag{4.5}$$
$$c_2(\Omega) = (1 + b)(1 + \alpha), \tag{4.6}$$
$$a_2(\Omega) = -(1 + b)^2\alpha, \tag{4.7}$$
$$|\Omega| = \frac{(1 + b)(b\alpha - 2)(b\alpha - 1 - \alpha)}{(1 + \alpha)}. \tag{4.8}$$

Proof. Observe $b < -1$ by Lemma 3.3 and Γ contains no parallelograms of length 3 by Theorem 1.1. Hence there exists a weak-geodetically closed subgraph Ω of diameter 2 by Theorem 4.2. By applying (2.17), (2.18) and (2.20) to (4.1)-(4.3), we have (4.4)-(4.7) immediately. Observe that $|\Omega| = 1 + k(\Omega) + k(\Omega)b_1(\Omega)/c_2(\Omega)$. (4.8) follows from this and (4.4)-(4.6). □

Proposition 4.4. *[9, Proposition 3.2] Let Γ denote a distance-regular graph with diameter $d \geq 3$. Suppose there exists a weak-geodetically closed subgraph Ω of Γ with diameter 2. Then the intersection numbers of Γ satisfy the following inequality*

$$a_3 \geq a_2(c_2 - 1) + a_1. \tag{4.9}$$

Corollary 4.5. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Suppose the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then*

$$c_2 \leq b^2 + b + 2. \tag{4.10}$$

Proof. Applying $a_1 = 0$ in (2.20), we have $a_3 = -\alpha(b^2 + b + 1)(b + 1)^2$. Then by applying (4.9) using Lemma 3.3, (4.1), (4.7), the result follows immediately. □

We will decrease the upper bound of c_2 in (4.10). We need the following lemma.

Lemma 4.6. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume the intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Let Ω be a weak-geodetically closed subgraph of diameter 2 in Γ. Let $r > s$ denote the nontrivial eigenvalues of the strongly regular graph Ω. Then the following (i), (ii) hold:*

(i) The multiplicity of r is

$$f = \frac{(b\alpha - 1)(b\alpha - 1 - \alpha)(b\alpha - 1 + \alpha)}{(\alpha - 1)(\alpha + 1)}. \tag{4.11}$$

(ii) The multiplicity of s is

$$g = \frac{-b(b\alpha - 1)(b\alpha - 2)}{(\alpha - 1)(\alpha + 1)}. \tag{4.12}$$

Proof. From [3, Theorem 21.1], we have

$$f = \frac{1}{2}\{v - 1 + \frac{(v-1)(c_2 - a_1) - 2k}{\sqrt{(c_2 - a_1)^2 + 4(k - c_2)}}\}, \tag{4.13}$$

$$g = \frac{1}{2}\{v - 1 - \frac{(v-1)(c_2 - a_1) - 2k}{\sqrt{(c_2 - a_1)^2 + 4(k - c_2)}}\}, \tag{4.14}$$

where $v=|\Omega|$ and k is the valency of Ω. Note $c_2(\Omega) = (1+b)(1+\alpha)$ by (2.17), $k(\Omega) = (1 + b)(1 - \alpha b)$ by (4.4), and $v = (1 + b)(b\alpha - 2)(b\alpha - 1 - \alpha)/(1 + \alpha)$ by (4.8). Now (4.11), (4.12) follow from (4.13), (4.14). □

Corollary 4.7. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then*

$$\frac{b(b+1)^2(b+2)}{c_2}, \tag{4.15}$$

$$\frac{(b-2)(b-1)b(b+1)}{2+2b-c_2} \tag{4.16}$$

are both integers.

Proof. Let f and g be as (4.11), (4.12). Set $\rho{=}\alpha(1+b)$. Note ρ is an integer, since $\rho = c_2 - 1 - b$. Then both

$$f + g - (1 - 3b^2 - b\rho + b^2\rho - b^3) = \frac{2b + 5b^2 + 4b^3 + b^4}{1 + b + \rho} = \frac{b(b+1)^2(b+2)}{c_2}$$

and

$$f - g - (1 - 3b^2 - b\rho + b^2\rho + b^3) = \frac{2b - b^2 - 2b^3 + b^4}{-1 - b + \rho} = \frac{(b-2)(b-1)b(b+1)}{2+2b-c_2}$$

are integers since f, g, b and ρ are integers. $\square$

Proposition 4.8. *Let Γ denote a distance-regular graph with classical parameters (d, b, α, β), where $d \geq 3$. Assume Γ has intersection numbers $a_1 = 0$ and $a_2 \neq 0$. Then $c_2 \leq b(b+1)$.*

Proof. Recall $c_2 \leq b^2 + b + 2$ by (4.10). First, suppose

$$c_2 = b^2 + b + 2. \tag{4.17}$$

Then the integral condition (4.15) becomes

$$b^2 + 3b + \frac{-4b}{b^2 + b + 2}. \tag{4.18}$$

Since $0 < -4b < b^2 + b + 2$ for $b \leq -5$, we have $-4 \leq b \leq -2$. For $b = -4$ or -3, expression (4.18) is not an integer. The remaining case $b{=}-2$ implies $\alpha = -5$ by (4.6), $v = 28$ by (4.8) and $g = 6$ by (4.12). It contradicts to $v \leq \frac{1}{2}g(g+3)$ [3, Theorem 21.4]. Hence $c_2 \neq b^2 + b + 2$. Next suppose $c_2 = b^2 + b + 1$. Then (4.16) becomes

$$-b^2 + b + 1 + \frac{1}{b^2 - b - 1}. \tag{4.19}$$

It fails to be an integer since $b < -1$. $\square$

Proof of Theorem 1.2:

The results come from Corollary 4.7 and Proposition 4.8. □

Example 4.9. [4] Hermitian forms graph $Her_2(d)$ is a distance-regular graph with classical parameters (d, b, α, β) with $b = -2$, $\alpha = -3$ and $\beta = -((-2)^d + 1)$, which satisfies $a_1 = 0$, $a_2 \neq 0$ and $c_2 = b(b+1)$.

Example 4.10. [3, p. 237] Gerwitz graph is a distance-regular graph with diameter 2 and intersection numbers $a_1 = 0$, $c_2 = 2$, $k = 10$, which can be written as classical parameters (d, b, α, β) with $d = 2$, $b = -3$, $\alpha = -2$, $\beta = -5$, so we have $c_2 = \dfrac{(b+1)^2}{2}$.

Conjecture 4.11. *(Gerwitz graph does not grow.) There is no distance-regular graph with classical parameters* $(d, -3, -2, -\dfrac{1+(-3)^d}{2})$*, where* $d \geq 3$.

There is a similar conjecture of Conjecture 4.11 for the complement part in $a_1 \neq 0$. See [10, Theorem 10.3] for details.

Acknowledgement

The authors thank Paul Terwilliger for reading the first manuscript and giving valuable suggestions.

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