A Typical Vertex of a Tree

Wei, Ping-hong* Weng, Chih-wen†

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Abstract

Let T denote a tree with at least three vertices. Observe that T contains a vertex which has at least two neighbors of degree one or two. A class of algorithms on trees related to the observation are discussed and characterized. One of the example is an algorithm to compute the minimum rank $m(T)$ of the symmetric matrices with prescribed graph T, which is easier to process than the algorithm previous found by P. Nylen[Linear Algebra Appl. 248:303-316(1996)]. Two interpretations of the number $m(T)$ in terms of some combinatorial properties on trees are given.

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1 Introduction and results.

Let T denote a tree with $n(T)$ vertices. We also use T as its vertex set. We refer the reader to [3, p376-p388] for the definition and the properties of trees. For a vertex subset $U \subseteq T$, let $T \setminus U$ denote the subgraph induced on the vertex subset $T \setminus U$ of T. Let p be a vertex of T, and let $T_p^1, \cdots, T_p^t$

*Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C.

†Department of Applied Mathematics, National Chiao Tung University, Taiwan R.O.C.

denote the connected components of $T \setminus \{p\}$. Note that each T_p^i is a tree. Observe

$$n(T) = n(T_p^1) + \cdots + n(T_p^t) + 1. \tag{1}$$

Let P_n denote the simple path with n vertices. Line (1) can be viewed as a trivial algorithm on trees to compute $n(T)$ provided the initial condition $n(P_1) = 1$. The choice of a vertex p does not affect the value $n(T)$.

We shall give another algorithm on trees. We need a few definitions first. For an $n \times n$ symmetric matrix $A = [a_{ij}]$, we associate with it the graph $\Gamma(A)$ having n vertices labeled $1, 2, \ldots, n$. For $i \neq j$, the unordered pair (i, j) will be an edge in $\Gamma(A)$ if and only if $a_{ij} \neq 0$. Given a graph G on n vertices, we define the number $m(G)$ by

$$m(G) := \min\{\text{rank } A \mid \Gamma(A) = G\}. \tag{2}$$

The study of $m(G)$ can be found in [1], [2], [4]. Observe

$$m(P_1) = 0, m(P_2) = 1. \tag{3}$$

A vertex p of T is called *appropriate* if at least two of the connected components in $T \setminus \{p\}$ are the simple paths (one or more vertices) which were connected to p through an endpoint. It is not difficult to see that every tree T with at least 3 vertices has an appropriate vertex. See [1, Lemma 3.1] for details. Provided the initial conditions in (3), P. Nylen[1] gives the algorithm

$$m(T) = m(T_p^1) + \cdots + m(T_p^t) + 2 \tag{4}$$

to compute $m(T)$, where $n(T) \geq 3$ and p is an appropriate vertex of T. The choice of p among the appropriate vertices of T does not affect the number $m(T)$ also.

Motivated by the above definition, we define a vertex p of T to be *typical* if p has at least two neighbors of degrees 1 or 2 in T. It is immediate from the definition that an appropriate vertex is a typical vertex. In Figure 1, the vertices labeled 2, 4, 6, 11 are typical and only the vertices labeled 2, 11 are appropriate.

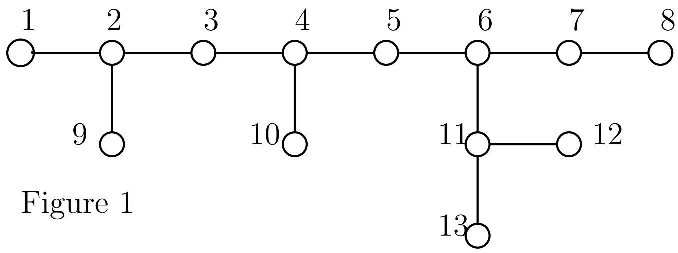

Figure 1

We shall prove in Theorem 1.7 that the condition p being appropriate in line (4) can be replaced by p being typical. We study a general class of algorithms on trees first. Fix three reals a, b, c. We assign a tree T with the real numbers $f(T)$ recursively by the following rules:

$$f(P_1) = a, f(P_2) = b, \tag{5}$$

$$f(T) = f(T_p^1) + \cdots + f(T_p^t) + c, \tag{6}$$

where p is a typical vertex of T. Note that $f(T)$ may not have a unique solution, since the choice of a typical vertex p may be different. For $a = 1$, $b = 2$, $c = 1$, $f = n$, (5)-(6) is the case of (1) with p typical. We list our results in this section and the proofs shall be in next section.

Lemma 1.1. Suppose the algorithm in (5)-(6) generates a unique solution $f(T)$ for each tree T. Then $3a - 2b + c = 0$.

We shall prove the converse of Lemma 1.1 in Theorem 1.4. In fact, if $3a - 2b + c = 0$ then we can express $f(T)$ into a linear combination of $n(T)$ and the number $s(T)$ defined below. For a vertex subset $U \subseteq T$, let $c_T(U)$ denote the number of connected components in the subgraph $T \setminus U$. The *separating number* of a tree T is the number

$$s(T) := \max\{c_T(U) - |U| \ \mid \ U \subseteq T\}. \tag{7}$$

U is a *separating* set of T if $c_T(U) - |U| = s(T)$. Note that if U is a separating set of T, $T \setminus U$ is a union of simple paths. Observe

$$s(P_1) = 1, s(P_2) = 1. \tag{8}$$

Theorem 1.2 gives an algorithm to construct a separating set, and to determine the separating number of a tree.

Theorem 1.2. Let T be a tree with at least 3 vertices and p be a typical vertex of T. Let $T_p^1, \ldots, T_p^t$ be the connected components of $T \setminus \{p\}$. Let U be a subset of vertices of T containing p. Then U is a separating set of T if and only if for each $i \quad (1 \leq i \leq t)$, $U \cap T_p^i$ is a separating set of T_p^i. Furthermore,

$$s(T) = s(T_p^1) + \cdots + s(T_p^t) - 1. \tag{9}$$

Note that (8)-(9) is the case $a = 1$, $b = 1$, $c = -1$ and $f = s$ of (5)-(6). It follows from (8)-(9) that $s(P_n) = 1$. Corollary 1.3 improves the algorithm in Theorem 1.2.

Corollary 1.3. Let U be a subset of the typical vertices of T satisfying the following (*) condition of T:

(*) Each vertex of U with degree 2 in T is not adjacent to other vertices in U.

Let $T_U^1, \cdots, T_U^l$ be the connected components of $T \setminus U$. Suppose S_j is a separating set of $T_U^j \quad (1 \leq j \leq l)$. Then

$$U \cup \Big(\bigcup_{1 \leq j \leq l} S_j \Big)$$

is a separating set of T. Furthermore,

$$s(T) = s(T_U^1) + \cdots + s(T_U^l) - |U|. \tag{10}$$

The following theorem shows that $n(T)$ and $s(T)$ span all the functions defined on trees satisfying (5)-(6).

Theorem 1.4. Suppose $3a - 2b + c = 0$. Then $f(T)$ are numbers generated from (5)-(6) for trees T if and only if

$$f(T) = \frac{a + c}{2} n(T) + \frac{a - c}{2} s(T) \tag{11}$$

for trees T. In particular, $f(T)$ has a unique solution for each tree T.

For graph theoretical interest, we give another interpretation of $s(T)$ in Corollary 1.6. Let $e(T)$ denote the number of edges in T. Note that $e(T) = n(T) - 1$. A subset F of the edge set $E(T)$ of T *dissolves* the tree T if the subgraph $T \setminus F$ obtained from T by deleting all edges in F is a disjoint union of simple paths. Set

$$s^*(T) := \min\{|F| \mid F \subseteq E(T) \text{ dissolves } T\}. \tag{12}$$

An edge subset F is a *separating* edge set of T if F dissolves T and $|F| = s^*(T)$. Observe $s^*(P_n) = 0$.

Theorem 1.5. Let T be a tree with at least 3 vertices and p be a typical vertex of degree t. Let $e_1, \cdots, e_t$ denote the edges incident on p, and $T_p^1, \ldots, T_p^t$ the connected components of $T \setminus \{p\}$. Assume each of e_{t-1}, e_t is incident on a vertex different from p of degree at most 2 in T. Suppose F_i is a separating edge set of $T_p^i \quad (1 \leq i \leq t)$. Then

$$\{e_1, \cdots, e_{t-2}\} \cup \bigcup_{1 \leq i \leq t} F_i$$

is a separating edge set of T. Furthermore,

$$s^*(T) = s^*(T_p^1) + \cdots + s^*(T_p^t) + t - 2. \tag{13}$$

Equivalently, $g(T) := e(T) - s^*(T)$ satisfies

$$g(T) = g(T_p^1) + \cdots + g(T_p^t) + 2. \tag{14}$$

Corollary 1.6.

$$s(T) = s^*(T) + 1. \tag{15}$$

Theorem 1.7. Let T be a tree with at least 3 vertices and p be a typical vertex of degree t. Let $T_p^1, \ldots, T_p^t$ be the connected components of $T \setminus \{p\}$. Then

$$m(T) = m(T_p^1) + \cdots + m(T_p^t) + 2, \tag{16}$$

where $m(T)$ is defined in (2).

Following the above lines, we reprove the following Corollary which was proved by C. R. Johnson and A.L. Duarte[5].

Corollary 1.8. $m(T) = e(T) - s^*(T) = n(T) - s(T)$.

To end this section, we show how to compute $m(T)$ for the tree T in Figure 1. The best algorithm is corollary 1.3. We set $U = \{2, 4, 6, 11\}$ which of course satisfies (*) condition of Corollary 1.3. Since $T \setminus U$ contains 8 simple paths, the separating number $s(T) = 8 - 4 = 4$ by (10). Now $m(T) = 13 - 4 = 9$ by Corollary 1.8.

2 Proofs of results.

Proof of Lemma 1.1. Suppose the algorithm in (5)-(6) generates a unique solution $f(T)$ for each tree T. Considering the simple path P_3 of three vertices, the middle vertex is typical, so $f(P_3) = 2a + c$ by (5)-(6). For the simple path P_5 of five vertices, there are essentially two different ways to choose a typical vertex. According to these two ways,

$$\begin{aligned} f(P_5) &= f(P_2) + f(P_2) + c \\ &= 2b + c, \end{aligned}$$

and

$$\begin{aligned} f(P_5) &= f(P_1) + f(P_3) + c \\ &= a + (2a + c) + c. \end{aligned}$$

Hence $3a - 2b + c = 0$.

Proof of Theorem 1.2. We find an upper bound of $s(T)$ first. Let V denote a vertex subset of T. We shall prove

$$c_T(V) - |V| \leq s(T_p^1) + \cdots + s(T_p^t) - 1. \tag{17}$$

Set $V_i = V \cap T_p^i \quad (1 \leq i \leq t)$. Suppose $p \in V$. Then

$$|V| = 1 + \sum_{i=1}^{t} |V_i|, \tag{18}$$

and the components in $T \setminus V$ are exactly those in $T_p^i \setminus V_i \quad (1 \leq i \leq t)$. Hence

$$\begin{aligned} c_T(V) - |V| &= \sum_{i=1}^{t} c_{T_p^i}(V_i) - (1 + \sum_{i=1}^{t} |V_i|) \\ &= \sum_{i=1}^{t} (c_{T_p^i}(V_i) - |V_i|) - 1 \\ &\leq s(T_p^1) + \cdots + s(T_p^t) - 1. \end{aligned} \tag{19}$$

Suppose $p \notin V$. Then

$$|V| = \sum_{i=1}^{t} |V_i|. \tag{20}$$

Let u denote the number of neighbors of p in $T \setminus V$. Each of the u vertices is in a connected component of $T_p^i \setminus V_i$ which contains it, and p merges these u components into a single connected component of $T \setminus V$. Then

$$c_T(V) = 1 - u + \sum_{i=1}^{t} c_{T_p^i}(V_i). \tag{21}$$

Let v denote the number of neighbors of p in V which have degrees 1 or 2 in T. Since each of these v vertices has degree 0 or 1 in the subgraph T_p^i which contains it, and by the fact, a separating set contains no endpoints, we have the corresponding V_i is not a separating set of T_p^i. Hence there are at least v indices i such that

$$c_{T_p^i}(V_i) - |V_i| + 1 \leq s(T_p^i).$$

Then

$$v + \sum_{i=1}^{t} (c_{T_p^i}(V_i) - |V_i|) \leq \sum_{i=1}^{t} s(T_p^i). \tag{22}$$

Note that

$$u + v \geq 2, \tag{23}$$

since p is typical. Then by (20)-(23),

$$\begin{aligned} c_T(V) - |V| &= 1 - u + \sum_{i=1}^{t} c_{T_p^i}(V_i) - \sum_{i=1}^{t} |V_i| \\ &= 1 - u + \sum_{i=1}^{t} (c_{T_p^i}(V_i) - |V_i|) \\ &\leq s(T_p^1) + \cdots + s(T_p^t) + 1 - u - v \\ &\leq s(T_p^1) + \cdots + s(T_p^t) - 1. \end{aligned} \tag{24}$$

This proves (17). To prove Theorem 1.2, set $V = U$ in (17). Then $p \in V$. Suppose $V_i = V \cap T_p^i$ is a separating set of T_p^i for all i. Then equality holds in (19). Hence for the vertex set V, $c_T(V) - |V|$ attains its maximum in (17). We conclude V is separating set of T, and (9) holds. To prove the other direction, suppose V is a separating set of T. Then equality holds in (17) and (19). This forces

$$c_{T_p^i}(V_i) - |V_i| = s(T_p^i) \quad (1 \leq i \leq t),$$

where $V_i = V \cap T_p^i$. Hence for each $i \quad (1 \leq i \leq t)$, $V \cap T_p^i$ is a separating set of T_p^i. This proves the theorem.

Proof of Corollary 1.3. We prove the corollary by induction on the cardinality of U. This is clear if U is empty. Assume U is not empty. Pick $p \in U$. Let $T_p^1, \cdots, T_p^t$ denote the connected components of $T \setminus \{p\}$. Fix an integer $i \quad (1 \leq i \leq t)$. Observe that T_p^i contains those T_U^j it intersects. First we prove that

$$(U \cap T_p^i) \cup (\bigcup_{S_j \subseteq T_p^i} S_j) \tag{25}$$

is a separating set of T_p^i, and

$$s(T_p^i) = \sum_{T_U^j \subseteq T_p^i} s(T_U^j) - |U \cap T_p^i|. \tag{26}$$

(25)-(26) follow from induction, if we prove $U \cap T_p^i$ contains typical vertices of T_p^i satisfying (*) condition of T_p^i. Let x denote the neighbor of p in T_p^i. Note that for vertices in T_p^i, the degrees in T and the degrees in T_p^i are the same except the vertex x whose degrees are decreased by 1. Hence we only need to show that if $x \in U$ then x is also typical in T_p^i, and furthermore, if x has degree 2 in T_p^i then x is not adjacent to other vertices in $U \cap T_p^i$. Suppose $x \in U$. Then p has degree at least 3, since U satisfies the (*) condition of T. Hence x is also typical in T_p^i by the definition of typical. Furthermore, suppose x has degree 2 in T_p^i. By the definition of typical again, the two neighbors of x in T_p^i have degrees 1 or 2 in T, and then are not contained in U since U satisfies the (*) condition of T. This proves (25)-(26). By applying Theorem 1.2 to (25)-(26),

$$\begin{aligned} & \{p\} \cup \bigcup_{1\leq i\leq t} ((U \cap T_p^i) \cup (\bigcup_{S_j \subseteq T_p^i} S_j)) \\ = \; & U \cup (\bigcup_{1\leq j\leq l} S_j) \end{aligned}$$

is a separating set of T, and

$$\begin{aligned} s(T) &= s(T_p^1) + \cdots + s(T_p^t) - 1 \\ &= \sum_{1\leq i\leq t} (\sum_{T_U^j \subseteq T_p^i} s(T_U^j) - |U \cap T_p^i|) - 1 \\ &= s(T_U^1) + \cdots + s(T_U^l) - |U|. \end{aligned}$$

This proves the corollary.

Proof of Theorem 1.4. First assume $f(T)$ are numbers generated from (5)-(6). We prove by induction on the number $n(T)$. Note that $n(P_1) = 1$, $n(P_2) = 2$, $s(P_1) = s(P_2) = 1$, $f(P_1) = a$, $f(P_2) = b$. Hence (11) can be checked directly if $n(T) \leq 2$. Assume $n(T) \geq 3$. Pick a typical vertex p in T. By (6), induction, (1) and (9), we obtain

$$\begin{aligned} f(T) &= f(T_p^1) + \cdots + f(T_p^t) + c \\ &= \frac{a+c}{2}\sum_{i=1}^{t} n(T_p^i) + \frac{a-c}{2}\sum_{i=1}^{t} s(T_p^i) + c \\ &= \frac{a+c}{2}(\sum_{i=1}^{t} n(T_p^i) + 1) + \frac{a-c}{2}(\sum_{i=1}^{t} s(T_p^i) - 1) \\ &= \frac{a+c}{2} n(T) + \frac{a-c}{2} s(T). \end{aligned} \tag{27}$$

This proves the necessary condition (11). $f(T)$ has a unique solution, since $n(T)$, $s(T)$ in (11) are well-defined functions. For the other direction, we assume (11) holds. (5) can be check directly. Reversing above four equalities in (27), we obtain $f(T)$ satisfies (6). This proves the theorem.

Proof of Theorem 1.5. We give a lower bound of $s^*(T)$ first. Suppose $F' \subseteq E(T)$ dissolves T. We shall prove

$$|F'| \geq s^*(T_p^1) + \cdots + s^*(T_p^t) + t - 2. \tag{28}$$

Set $F_i' = F' \cap E(T_p^i) \quad (1 \leq i \leq t)$. Since the vertex p has degree t in T, and $T \setminus F'$ are simple paths, F' contains at least $t-2$ edges incident on p. Hence

$$|F'| \geq |F_1'| + \cdots + |F_t'| + t - 2. \tag{29}$$

Observe that F_i' dissolves T_p^i. Hence

$$|F_i'| \geq s^*(T_p^i) \quad (1 \leq i \leq t). \tag{30}$$

(28) follows from (29)-(30). To prove the theorem, set

$$F' = \{e_1, \cdots, e_{t-2}\} \cup (\bigcup_{1 \leq i \leq t} F_i).$$

Hence $F_i' = F_i$. Observe F' dissolves T, and equalities hold in (29)-(30). Hence equality holds in (28). This proves that (13) holds and F' is a separating edge set of T. To prove (14), observe

$$\begin{aligned} g(T) &= e(T) - s^*(T) \\ &= e(T) - s^*(T_p^1) - \cdots - s^*(T_p^t) - t + 2 \\ &= \sum_{1 \leq i \leq t} (e(T_p^i) - s^*(T_p^i)) + 2 \\ &= \sum_{1 \leq i \leq t} g(T_p^i) + 2. \end{aligned}$$

Proof of Corollary 1.6. With the notation of Theorem 1.5, observe $g(P_n) = e(P_n) - s^*(P_n) = n - 1$, especially $g(P_1) = 0$ $g(P_2) = 1$. Hence (14) is the case $f = g$, $a = 0$, $b = 1$, and $c = 2$ in (5)-(6). We obtain $e(T) - s^*(T) = n(T) - s(T)$ by (11). Then $s(T) = s^*(T) + 1$, since $n(T) - e(T) = 1$.

Proof of Theorem 1.7. $m(T)$ is the unique solution of the algorithm in (3)-(4). However (3)-(4) is a special case of (5)-(6) with p appropriate, $a = 0$, $b = 1$ and $c = 2$. Since $3a - 2b + c = 0$, the algorithm in (5)-(6) with p typical has the unique solution $m(T)$ by Theorem 1.4.

Proof of Corollary 1.8. The result follows by applying (3), (16) to (11) using (15).

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