Weak-geodetically Closed Subgraphs in Distance-Regular Graphs

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Abstract. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 2$ and distance function δ. A (vertex) subgraph $\Omega \subseteq X$ is said to be *weak-geodetically closed* whenever for all $x, y \in \Omega$ and all $z \in X$,

$$\delta(x,z) + \delta(z,y) \leq \delta(x,y) + 1 \quad \longrightarrow \quad z \in \Omega.$$

We show that if the intersection number $c_2 > 1$ then any weak-geodetically closed subgraph of X is distance-regular. Γ is said to be *i-bounded*, whenever for all $x, y \in X$ at distance $\delta(x,y) \leq i$, x, y are contained in a common weak-geodetically closed subgraph of Γ of diameter $\delta(x,y)$. By a *parallelogram of length i*, we mean a 4-tuple $xyzw$ of vertices in X such that $\delta(x,y) = \delta(z,w) = 1$, $\delta(x,w) = i$, and $\delta(x,z) = \delta(y,z) = \delta(y,w) = i-1$. We prove the following two theorems.

Theorem 1. Let Γ denote a distance-regular graph with diameter $D \geq 2$, and assume the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Then for each integer $i \ \ (1 \leq i \leq D)$, the following (i)-(ii) are equivalent.

(i) Γ is i-bounded.
(ii) Γ contains no parallelogram of length $\leq i+1$.

Restricting attention to the Q-polynomial case, we get the following stronger result.

Theorem 2. Let Γ denote a distance-regular graph with diameter $D \geq 3$, and assume the intersection numbers $c_2 > 1$, $a_1 \neq 0$. Suppose Γ is Q-polynomial. Then the following (i)-(iii) are equivalent.

(i) Γ contains no parallelogram of length 2 or 3.
(ii) Γ is D-bounded.
(iii) Γ has classical parameters (D, b, α, β), and either $b < -1$, or else Γ is a dual polar graph or a Hamming graph.

1. Introduction.

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 2$, and let δ denote the distance function of Γ.

Recall a (vertex) subgraph $\Omega \subseteq X$ is *geodetically closed* whenever for all vertices $x, y \in \Omega$, and for all vertices $z \in X$,

$$\delta(x,z) + \delta(z,y) = \delta(x,y) \longrightarrow z \in \Omega.$$

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Distance-regular graphs containing many geodetically closed subgraphs have been studied by several authors. Shult and Yanushka[4], Brouwer and Wilbrink[3] showed that if Γ is a near polygon with $c_2 > 1$, $a_1 \neq 0$, then there exist sub $2j$-gons in Γ for each integer $j \quad (2 \leq j \leq D)$. Also, Ivanov and Shpectorov[5] showed that if Γ is a Hermitian forms graph, then Γ has geodetically closed subgraphs of any diameter $j \;\; (1 \leq j \leq D)$.

In the present paper, we study the following special kind of geodetically closed subgraphs. We say a subgraph $\Omega \subseteq X$ is *weak-geodetically closed* whenever for all vertices $x, y \in \Omega$, and for all $z \in X$,

$$\delta(x,z) + \delta(z,y) \leq \delta(x,y) + 1 \longrightarrow z \in \Omega.$$

We have two main results. First, given an integer $i \quad (1 \leq i \leq D)$, we give necessary and sufficient conditions for the existence of a weak-geodetically closed subgraph of diameter $\delta(x,y)$ containing any two given vertices x, y with $\delta(x,y) \leq i$. Theorem 6.4 is our main result in this area.

We then tighten Theorem 6.4 in the case Γ is Q-polynomial, and obtain our second main result, Theorem 7.2.

The paper is organized as follows. In sections 2-5, we set up the necessary tools for the proof of Theorem 6.4. To do this, we study the structure theory of a weak-geodetically closed subgraph Ω of Γ.

More precisely, in section 2, we define the notion of a subgraph being *weak-geodetically closed with respect to a vertex.* We find necessary and sufficient conditions for a subgraph to be weak-geodetically closed with respect to some vertex.

In section 3, we get some inequalities involving the intersection numbers of Γ, when we assume the existence of certain weak-geodetically closed subgraphs. Proposition 3.2 is the main result in this section.

In section 4, we consider a regular connected subgraph Ω of Γ. First, we find a lower bound for $\left|\Omega\right|$ and necessary and sufficient conditions for this bound to be met(Lemma 4.4). These conditions involve the notion of weak-geodetic closure. In our main result of this section, Theorem 4.6, we show Ω is weak-geodetically closed if and only if Ω is weak-geodetically closed with respect to at least one vertex.

In section 5, we restrict to the case $c_2 > 1$, and prove a weak-geodetically closed subgraph Ω of Γ is distance-regular.

We prove the two main theorems in section 6 and section 7.

For the rest of this section, we give some definitions.

Let $\Gamma = (X, R)$ be a finite undirected graph without loops or multiple edges, with vertex set X and edge set R. We say vertices x, y are *adjacent* if $xy \in R$. Pick any integer $i \ (0 \leq i \leq D)$ and any vertices $x, y \in X$. By a *path of length* i from x to y, we mean a sequence $x = x_0, x_1, \cdots, x_i = y$ of vertices from x such that x_j, x_{j+1} are adjacent for all $j \ \ (0 \leq j \leq i-1)$. Being joined by a path is an equivalence relation. Its equivalence classes are called the *connected components* of Γ. Γ is said to be connected whenever Γ has a unique connected component. From now on, assume Γ is connected. The *distance* $\delta(x, y)$ between two vertices $x, y \in X$ is the length of a shortest (*geodesic*) path from x to y. By the *diameter* of Γ, we mean the scalar

$$D := \max\{\delta(x, y) | x, y \in X\}.$$

Sometimes we write $\mathrm{diam}(\Gamma)$ to denote the diameter of Γ. By a *clique* in Γ, we mean a set of mutually adjacent vertices in X.

Let $\Gamma = (X, R)$ be a graph with diameter D. By a *subgraph* of Γ, we mean a graph (Ω, Ξ), where Ω is a nonempty subset of X and $\Xi = \{\ xy \ | x, y \in \Omega,\ \ xy \ \in R\}$. We refer to (Ω, Ξ) as the subgraph *induced on* Ω and by abuse of notation, we refer to this subgraph as Ω. For any $x \in X$ and any integer i, set

$$\Gamma_i(x) := \{y | y \in X, \delta(x, y) = i\},$$

and for $y \in \Gamma_i(x)$, set

$$B(x, y) := \Gamma_1(x) \cap \Gamma_{i+1}(y), \tag{1.1}$$

$$A(x, y) := \Gamma_1(x) \cap \Gamma_i(y), \tag{1.2}$$

$$C(x, y) := \Gamma_1(x) \cap \Gamma_{i-1}(y). \tag{1.3}$$

Note that for all $x, y \in \Gamma$ and for all $z \in C(y, x)$, we have

$$C(x, z) \subseteq C(x, y), \tag{1.4}$$

$$B(x, z) \supseteq B(x, y). \tag{1.5}$$

The *valency* $k(x)$ of a vertex $x \in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k. Γ is said to be *distance-regular* whenever for all integers $i \ \ (0 \leq i \leq D)$, and for all $x, y \in X$ with $\delta(x, y) = i$, the numbers

$$c_i := \Big|C(x, y)\Big|, \tag{1.6}$$

$$a_i := \Big|A(x, y)\Big|, \tag{1.7}$$

$$b_i := \Big|B(x, y)\Big| \tag{1.8}$$

are independent of x, y. The constants $c_i, a_i, b_i \quad (0 \leq i \leq D)$ are known as the *intersection numbers* of Γ. The sequence

$$\{b_0,\ b_1, \cdots, b_{D-1};\ c_1,\ c_2, \cdots,\ c_D\}$$

is called the *intersection array* of Γ. Note that the valency $k = b_0$, $c_0 = 0$, $c_1 = 1$, $b_D = 0$, and

$$k = c_i + a_i + b_i \qquad (0 \leq i \leq D) \tag{1.9}$$

[2, 126].

2. Weak-geodetically closed subgraphs with respect to a vertex.

Let $\Gamma = (X, R)$ denote a graph, and let Ω denote a subgraph of Γ. In this section, we define what it means for Ω to be *weak-geodetically closed with respect to a vertex*. We find some necessary and sufficient conditions for Ω to have this property.

We begin with a definition.

Definition 2.1. Let $\Gamma = (X, R)$ denote a graph with distance function δ. Fix a subgraph Ω of Γ, and pick any vertex $x \in \Omega$. Ω is said to be *geodetically closed with respect to* x (resp. *weak-geodetically closed with respect to* x), whenever for all $y \in \Omega$ and for all $z \in X$,

$$\delta(x, z) + \delta(z, y) = \delta(x, y) \longrightarrow z \in \Omega$$

$$(\text{resp.}\ \ \delta(x, z) + \delta(z, y) \leq \delta(x, y) + 1 \longrightarrow z \in \Omega).$$

Lemma 2.2. Let $\Gamma = (X, R)$ denote a graph with distance function δ. Fix a subgraph Ω of Γ, and pick any vertex $x \in \Omega$. Then with the notation of (1.3), the following (i)-(iii) are equivalent.

(i) Ω is geodetically closed with respect to x.

(ii) $C(y, x) \subseteq \Omega$ for all $y \in \Omega$.

(iii) For all $y \in \Omega$, and for all $w \in \Gamma_1(y) \setminus \Omega$,

$$\delta(x, w) \geq \delta(x, y).$$

Proof. This is immediate from Definition 2.1.

Lemma 2.3. Let $\Gamma = (X, R)$ denote a graph with distance function δ. Fix a subgraph Ω of Γ, and pick any vertex $x \in \Omega$. Then with the notation of (1.2), (1.3), the following (i)-(iii) are equivalent.

(i) Ω is weak-geodetically closed with respect to x.

(ii) $C(y,x) \subseteq \Omega$ and $A(y,x) \subseteq \Omega$ for all $y \in \Omega$.

(iii) For all $y \in \Omega$, and for all $w \in \Gamma_1(y) \setminus \Omega$,

$$\delta(x,w) = \delta(x,y) + 1. \tag{2.1}$$

Proof. (i)$\longrightarrow$(ii). Let the vertex $y \in \Omega$ be given, and pick any $z \in A(y,x) \cup C(y,x)$. Then $\delta(x,z) \le \delta(x,y)$, and of course $\delta(z,y) = 1$, so

$$\delta(x,z) + \delta(z,y) \le \delta(x,y) + 1.$$

Hence $z \in \Omega$ by Definition 2.1.

(ii)$\longrightarrow$(iii). Let y, w be given. Observe

$$\begin{aligned} w &\in \Gamma_1(y) \setminus \Omega \\ &\subseteq B(y,x) \end{aligned}$$

by (ii), and (2.1) follows from (1.1).

(iii)$\longrightarrow$(i). Suppose Ω is not weak-geodetically closed with respect to x. Then by Definition 2.1, there exists a vertex $y \in \Omega$ and a vertex $z \notin \Omega$ such that

$$\delta(x,z) + \delta(z,y) \le \delta(x,y) + 1. \tag{2.2}$$

Of all such pairs y, z, pick one with $\delta(z,y)$ minimal. Note that $z \neq y$ by the construction, and $\delta(z,y) \neq 1$ by (2.1)-(2.2), so there exists a vertex $z' \in C(z,y)$. Observe

$$\delta(z',y) = \delta(z,y) - 1 \tag{2.3}$$

by the construction, and

$$\delta(x,z') \le \delta(x,z) + 1 \tag{2.4}$$

by the triangular inequality. Adding (2.2)-(2.4), we obtain

$$\delta(x,z') + \delta(z',y) \le \delta(x,y) + 1. \tag{2.5}$$

Observe $z' \in \Omega$ by (2.3), (2.5) and the construction. Now by (iii) (with $y := z', w := z$), we find

$$\delta(x,z) = \delta(x,z') + 1. \tag{2.6}$$

By the triangular inequality,

$$\delta(x,y) \le \delta(x,z') + \delta(z',y). \tag{2.7}$$

Adding (2.2), (2.3), and (2.7), we obtain

$$\delta(x,z) \leq \delta(x,z'),$$

contradicting (2.6). We conclude Ω is weak-geodetically closed with respect to x.

Lemma 2.4. Let $\Gamma = (X, R)$ denote a graph with distance function δ. Fix a subgraph Ω of Γ, and pick a vertex $x \in \Omega$. Suppose Ω is weak-geodetically closed with respect to x, and suppose there exists a vertex $z \in \Gamma_1(x) \setminus \Omega$. Then the following (i)-(ii) hold.

(i) For any vertex $y \in \Omega$,

$$\delta(z,y) = \delta(x,y) + 1.$$

(ii) x is the unique vertex in Ω adjacent to z.

Proof. (i). By Definition 2.1 and since $z \notin \Omega$, we have $\delta(x,z) + \delta(z,y) > \delta(x,y) + 1$. Of course $\delta(x,z) = 1$, so $\delta(z,y) > \delta(x,y)$. Also by the triangular inequality,

$$\begin{aligned}\delta(z,y) &\leq \delta(z,x) + \delta(x,y)\\ &= 1 + \delta(x,y).\end{aligned}$$

Hence $\delta(z,y) = \delta(x,y) + 1$.

(ii). This is immediate from (i).

Definition 2.5. Let $\Gamma = (X, R)$ denote a graph with distance function δ, and let Ω be any subgraph of Γ. For all vertices $x \in \Omega$, define

$$\text{diam}_x(\Omega) := \max\{\delta(x,y) | y \in \Omega\}.$$

Lemma 2.6. Let $\Gamma = (X, R)$ denote a distance-regular graph. Fix a subgraph Ω of Γ, and pick a vertex $x \in \Omega$. Suppose Ω is weak-geodetically closed with respect to x. Set $d :=\text{diam}_x(\Omega)$. Then the following (i)-(iv) hold.

(i) For all $y \in \Omega \cap \Gamma_d(x)$,

$$\Big|\Omega \cap \Gamma_1(y)\Big| = c_d + a_d.$$

(ii) For all $y \in \Omega \cap \Gamma_d(x)$,

$$C(x,y) \cup A(x,y) \subseteq \Omega \cap \Gamma_1(x). \tag{2.8}$$

(iii)

$$\left|\Omega \cap \Gamma_1(x)\right| \geq c_d + a_d.$$

(iv) Equality holds in (iii) if and only if equality holds in (2.8) for at least one $y \in \Omega \cap \Gamma_d(x)$, if and only if equality holds in (2.8) for all $y \in \Omega \cap \Gamma_d(x)$.

Proof. (i). Note that $\Gamma_1(y) = C(y,x) \cup A(y,x) \cup B(y,x)$. Observe that $\Omega \cap B(y,x) = \emptyset$ since $\delta(x,y) = \text{diam}_x(\Omega)$. Now $\Omega \cap \Gamma_1(y) = C(y,x) \cup A(y,x)$ by Lemma 2.3(ii), and (i) follows by (1.6), (1.7).

(ii). Pick $z \in C(x,y) \cup A(x,y)$. Then certainly $z \in \Gamma_1(x)$ and

$$\delta(x,z) + \delta(z,y) \leq \delta(x,y) + 1,$$

so $z \in \Omega$ by Definition 2.1.

(iii), (iv). These are immediate from (ii).

3. Weak-geodetically closed subgraphs.

Let $\Gamma = (X,R)$ be any graph. In this section, we study a subgraph Ω that is weak-geodetically closed with respect to all vertices in Ω. We prove that when Γ is distance-regular, the existence of Ω forces certain inequalities involving the intersection numbers of Γ.

Definition 3.1. Let $\Gamma = (X,R)$ be a graph. A subgraph Ω of Γ is said to be *geodetically closed* (resp. *weak-geodetically closed*) , whenever Ω is geodetically closed (resp. weak-geodetically closed) with respect to all $x \in \Omega$.

Note. A weak-geodetically closed subgraph Ω of Γ is geodetically closed in Γ. In particular, Ω is connected, and the distances as measured in Ω are the same as distances as measured in Γ.

Proposition 3.2. Let $\Gamma = (X,R)$ denote a distance-regular graph with diameter $D \geq 2$. Fix an integer d $(1 \leq d < D)$, and suppose there exists a weak-geodetically closed subgraph Ω of Γ that has diameter d. Then the intersection numbers b_i, a_i, c_i of Γ satisfy the following inequalities.

(i)

$$c_i \geq c_{i-1}(c_2 - 1) + 1 \qquad (1 \leq i \leq d+1).$$

(ii)

$$a_i \geq a_{i-1}(c_2 - 1) + a_1 \qquad (1 \leq i \leq d+1).$$

(iii)

$$b_i \leq (b_{i-1} - k)(c_2 - 1) + b_1 \qquad (1 \leq i \leq d+1).$$

(iv) Suppose $c_2 > 1$. Then

$$b_i < b_{i-1} \qquad (1 \leq i \leq d+1).$$

Proof. Let the integer i be given. Our result is clear if $i = 1$, since $c_1 = 1$, $c_0 = 0$, $a_0 = 0$, $b_0 = k$. Hence we may assume $i \geq 2$. First we claim there exist vertices $x, y \in \Omega$ and $z \in X \setminus \Omega$ such that

$$\delta(x,y) = i-1, \quad \delta(y,z) = 1, \quad \delta(x,z) = i. \tag{3.1}$$

Indeed, since $\mathrm{diam}(\Omega) = d$, we can pick vertices $x', y \in \Omega$ with $\delta(x', y) = d$. Observe $B(y, x') \neq \emptyset$ since $d < D$, so pick a vertex $z \in B(y, x')$. Note that $z \notin \Omega$, since

$$\begin{aligned} \delta(x', z) &= d+1 \\ &> \mathrm{diam}(\Omega). \end{aligned}$$

Now pick a vertex x in a geodesic path from x' to y with $\delta(x,y) = i-1$. Clearly, $x \in \Omega$, and x, y, z satisfy (3.1). This proves our claim. Recall by Lemma 2.4(ii),

$$\Gamma_1(z) \cap \Omega = \{y\}. \tag{3.2}$$

Now we consider the four parts of the proposition.

(i). Observe each vertex in $C(y,x)$ is adjacent to $c_2 - 1$ vertices in $C(z,x) \setminus \{y\}$. Next observe each vertex in $C(z,x) \setminus \{y\}$ is adjacent to at most 1 vertex in $C(y,x)$. To see this, pick any $w \in C(z,x) \setminus \{y\}$. Then $w \notin \Omega$ by (3.2). Note that $C(y,x) \subseteq \Omega$ by Lemma 2.3(ii), so w is adjacent to at most one vertex in $C(y,x)$ by Lemma 2.4(ii). Now by counting the edges between $C(z,x) \setminus \{y\}$ and $C(y,x)$, we find

$$\begin{aligned} c_i - 1 &= \Big|C(z,x) \setminus \{y\}\Big| \\ &\geq \Big|C(y,x)\Big|(c_2 - 1) \\ &= c_{i-1}(c_2 - 1), \end{aligned}$$

as desired.

(ii). We first prove

$$A(z,y) \subseteq A(z,x), \tag{3.3}$$

and then count the edges between $A(z,x) \setminus A(z,y)$ and $A(y,x)$ to establish the inequality.

Note that

$$A(y,x) \subseteq \Omega \tag{3.4}$$

by Lemma 2.3(ii),

$$A(z,y) \cap \Omega = \emptyset \tag{3.5}$$

by (3.2), and

$$A(z,y) \subseteq A(y,x) \cup A(z,x) \tag{3.6}$$

by construction. Now (3.3) follows from (3.4)-(3.6). We now count the edges between $A(z,x) \setminus A(z,y)$ and $A(y,x)$.

Claim 1. Each vertex in $A(z,x) \setminus A(z,y)$ is adjacent to at most one vertex in $A(y,x)$.

Proof of Claim 1. Observe that by (3.2),

$$A(z,x) \cap \Omega = \emptyset,$$

so Claim 1 follows from (3.4) and Lemma 2.4(ii).

Claim 2. Each vertex in $A(y,x)$ is adjacent to $c_2 - 1$ vertices in $A(z,x) \setminus A(z,y)$.

Proof of Claim 2. Pick $w \in A(y,x)$. Observe

$$w \in \Omega \tag{3.7}$$

by (3.4), so w is not adjacent to z by (3.2); in particular $\delta(w,z) = 2$. It now suffices to show

$$\Gamma_1(w) \cap (A(z,x) \setminus A(z,y)) = C(z,w) \setminus \{y\}, \tag{3.8}$$

since $\left| C(z,w) \setminus \{y\} \right| = c_2 - 1$. The inclusion

$$\Gamma_1(w) \cap (A(z,x) \setminus A(z,y)) \subseteq C(z,w) \setminus \{y\}$$

is clear by construction. To prove

$$C(z,w) \setminus \{y\} \subseteq \Gamma_1(w) \cap (A(z,x) \setminus A(z,y)),$$

pick $u \in C(z,w) \setminus \{y\}$. Of course $u \in \Gamma_1(w)$ and $u \in \Gamma_1(z)$, so

$$u \notin \Omega \tag{3.9}$$

by (3.2), and

$$u \in A(z,x) \cup A(w,x) \tag{3.10}$$

by construction. Note that

$$A(w,x) \subseteq \Omega \tag{3.11}$$

by (3.7) and Lemma 2.3(ii). Hence $u \in A(z,x)$ by (3.9)-(3.11). Also $u \notin A(z,y)$ by (3.7) and (3.9), otherwise u is adjacent to $y, w \in \Omega$, contradicting Lemma 2.4(ii). Hence we have (3.8). This proves Claim 2.

Now using Claim 1, Claim 2, we count the edges between $A(z,x) \setminus A(z,y)$ and $A(y,x)$, obtaining

$$\begin{aligned} a_i - a_1 &= \Big|A(z,x) \setminus A(z,y)\Big| \\ &\geq \Big|A(y,x)\Big|(c_2 - 1) \\ &= a_{i-1}(c_2 - 1), \end{aligned}$$

as desired.

(iii). By (i), (ii) and (1.9),

$$\begin{aligned} b_i &= k - a_i - c_i \\ &\leq k - (a_{i-1} + c_{i-1})(c_2 - 1) - a_1 - 1 \\ &= (b_{i-1} - k)(c_2 - 1) + b_1, \end{aligned}$$

as desired.

(iv). Observe $b_{i-1} - k \leq 0$, $c_2 - 1 \geq 1$ and $b_1 < k$, so by (iii),

$$\begin{aligned} b_i &\leq (b_{i-1} - k)(c_2 - 1) + b_1 \\ &\leq b_{i-1} - k + b_1 \\ &< b_{i-1}, \end{aligned}$$

as desired. This proves Proposition 3.2.

4. Regular subgraphs of distance-regular graphs.

In this section, we study basic properties of a regular connected subgraph Ω in a distance-regular graph, and get a lower bound of $\Big|\Omega\Big|$. We find necessary and sufficient conditions for $\Big|\Omega\Big|$ to meet this lower bound. These conditions are related to the weak-geodetically closed property. Theorem 4.6 is the main result of this section.

We begin with a definition.

Definition 4.1. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 2$, and let Ω denote a regular connected subgraph of Γ. We define

(i)
$$\beta_i(\Omega) := \gamma - c_i - a_i \qquad (0 \le i \le D),$$
where γ denotes the valency of Ω.

(ii)
$$k_i(\Omega) := \frac{\beta_0(\Omega)\beta_1(\Omega)\cdots\beta_{i-1}(\Omega)}{c_1 c_2 \cdots c_i} \qquad (1 \le i \le D),$$
$$k_0(\Omega) := 1.$$

(iii)
$$d(\Omega) := \min\{i | 0 \le i \le D, \beta_i(\Omega) \le 0\}. \tag{4.1}$$
(We observe $\beta_D(\Omega) \le 0$, so (4.1) makes sense).

Lemma 4.2. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \ge 2$. Let Ω denote a regular connected subgraph of Γ, and write $d := d(\Omega)$. Then the following (i)-(iii) hold.

(i) $\beta_i(\Omega) > 0 \qquad (0 \le i < d)$.

(ii) $k_i(\Omega) > 0 \qquad (0 \le i \le d)$.

(iii) $\gamma \le a_d + c_d$, where γ denotes the valency of Ω.

Proof. (i). This is immediate from Definition 4.1(iii).

(ii). This is immediate from (i) and Definition 4.1(ii).

(iii). This is immediate from Definition 4.1(i), (iii).

Lemma 4.3. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \ge 2$. Let Ω denote a regular connected subgraph of Γ, and pick any $x \in \Omega$. Pick an integer $i \quad (0 \le i \le d(\Omega))$. Then the following (i)-(iii) hold.

(i)
$$\Big|\Omega \cap \Gamma_i(x)\Big| \ge k_i(\Omega) \tag{4.2}$$
with equality if and only if
$$C(y, x) \subseteq \Omega \qquad (\forall y \in \Omega, \delta(x, y) \le i) \tag{4.3}$$
and
$$A(y, x) \subseteq \Omega \qquad (\forall y \in \Omega, \delta(x, y) \le i - 1). \tag{4.4}$$

(ii)
$$\Omega \cap \Gamma_i(x) \ne \emptyset. \tag{4.5}$$

(iii)
$$\mathrm{diam}_x(\Omega) \geq d(\Omega). \tag{4.6}$$

Proof. (i). We prove this by induction on the integer i. First assume $i = 0$. Then (4.2)-(4.4) hold at i; indeed both sides in (4.2) equal 1. Next assume $i \geq 1$. Then by Definition 4.1(i), a counting argument, the induction hypothesis and Definition 4.1(ii),

$$\begin{aligned} c_i\left|\Omega \cap \Gamma_i(x)\right| &\geq \text{number of edges between } \Omega \cap \Gamma_i(x) \text{ and } \Omega \cap \Gamma_{i-1}(x) && (4.7)\\ &\geq \beta_{i-1}(\Omega)\left|\Omega \cap \Gamma_{i-1}(x)\right| && (4.8)\\ &\geq \beta_{i-1}(\Omega)k_{i-1}(\Omega) && (4.9)\\ &= \frac{\beta_0(\Omega)\beta_1(\Omega)\cdots\beta_{i-1}(\Omega)}{c_1c_2\cdots c_{i-1}} && (4.10)\\ &= c_ik_i(\Omega), && (4.11)\end{aligned}$$

and equalities hold in (4.7)-(4.9) if and only if (4.3)-(4.4) hold. Now (4.2) follows since $c_i > 0$.

(ii). This is immediate from (i) above and Lemma 4.2(ii).

(iii). This is immediate from (ii) and Definition 2.5.

Lemma 4.4. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 2$, and let Ω denote a regular connected subgraph of Γ. Then

$$\left|\Omega\right| \geq k_0(\Omega) + k_1(\Omega) + \cdots + k_d(\Omega), \tag{4.12}$$

where $d := d(\Omega)$ is from Definition 4.1(iii). Furthermore, equality holds in (4.12) if and only if for any $x \in \Omega$ (and for some $x \in \Omega$),

$$d = \mathrm{diam}_x(\Omega), \tag{4.13}$$

$$C(y, x) \subseteq \Omega \qquad (\forall y \in \Omega, \delta(x, y) \leq d) \tag{4.14}$$

and

$$A(y, x) \subseteq \Omega \qquad (\forall y \in \Omega, \delta(x, y) \leq d - 1). \tag{4.15}$$

Proof. Pick $x \in \Omega$. Then by Lemma 4.3,

$$\begin{aligned} \left|\Omega\right| &= \sum_{i=0}^{\mathrm{diam}_x(\Omega)} \left|\Omega \cap \Gamma_i(x)\right| \\ &\geq \sum_{i=0}^{d} \left|\Omega \cap \Gamma_i(x)\right| && (4.16)\\ &\geq \sum_{i=0}^{d} k_i(\Omega), && (4.17)\end{aligned}$$

and equalities in (4.16)-(4.17) hold if (4.13)-(4.15) hold. Hence we have the lemma.

Theorem 4.5. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 2$. Let Ω denote a regular connected subgraph of Γ, and let $d := d(\Omega)$ be as in Definition 4.1(iii). Then the following (i)-(iii) are equivalent.

(i) Equality is obtained in (4.12).

(ii) Ω is geodetically closed, and for all $x, y \in \Omega$,

$$A(y, x) \subseteq \Omega \qquad \text{if } \delta(x, y) < \text{diam}_x(\Omega). \tag{4.18}$$

(iii) There exists a vertex $x \in \Omega$ such that

$$\Omega \text{ is geodetically closed with respect to } x, \tag{4.19}$$

and for all $y \in \Omega$,

$$A(y, x) \subseteq \Omega \qquad \text{if } \delta(x, y) < \text{diam}_x(\Omega). \tag{4.20}$$

If (i)-(iii) hold, then Ω is distance-regular, with diameter d, and intersection numbers

$$c_i(\Omega) = c_i \quad (0 \leq i \leq d), \tag{4.21}$$
$$a_i(\Omega) = a_i \quad (0 \leq i < d). \tag{4.22}$$

Proof. (i)→(ii) is immediate from Lemma 4.4, Lemma 2.2(ii), Definition 3.1 and (4.6). (ii)→(iii) is clear. To prove (iii)→(i), by Lemma 4.4, Lemma 2.2(ii), we only need to prove (4.13). Observe by a counting argument,

$$\Big|\Omega \cap \Gamma_i(x)\Big| c_i = \Big|\Omega \cap \Gamma_{i-1}(x)\Big| \beta_{i-1}(\Omega) \qquad (1 \leq i \leq \text{diam}_x(\Omega)),$$

forcing

$$\beta_i > 0 \qquad (0 \leq i < \text{diam}_x(\Omega)).$$

Hence (4.13) holds by (4.6) and Definition 4.1(iii).

Now suppose (i)-(iii) hold. (4.21)-(4.22) follow from (4.13) and (ii) above. We now have Theorem 4.5.

Theorem 4.6. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 2$. Let Ω denote a regular connected subgraph of Γ, and let $d := d(\Omega)$ be as in Definition 4.1(iii). Then the following (i)-(iii) are equivalent.

(i) Equality holds in (4.12), and Ω has valency $c_d + a_d$.

(ii) Ω is weak-geodetically closed.

(iii) Ω is weak-geodetically closed with respect to at least one vertex in Ω.

Suppose (i)-(iii) hold. Then Ω is distance-regular, with diameter d, and intersection numbers

$$c_i(\Omega) = c_i \qquad (0 \le i \le d), \tag{4.23}$$
$$a_i(\Omega) = a_i \qquad (0 \le i \le d). \tag{4.24}$$

Proof. Observe each of the three statements (i)-(iii) in the present theorem implies the corresponding statement in Theorem 4.5. Without loss of generality, we may assume Theorem 4.5(i)-(iii) hold. In particular, we may assume Ω is distance-regular with diameter d.

(i)→(ii). Since Theorem 4.5(ii) holds by assumption, it remains to show

$$A(y,x) \subseteq \Omega \tag{4.25}$$

for all $x, y \in \Omega$ such that $\delta(x,y) = d$. To obtain (4.25), observe by (4.21) that

$$\begin{aligned} \left| A(y,x) \setminus \Omega \right| &= a_d - a_d(\Omega) \\ &= a_d - (\left| \Omega \cap \Gamma_1(y) \right| - c_d) \\ &= 0, \end{aligned}$$

and (4.25) follows.

(ii)→(iii). This is clear.

(iii)→(i). Since Theorem 4.5(i) holds by assumption, it remains to show Ω has valency $c_d + a_d$. Pick any $x, y \in \Omega$ such that $\delta(x,y) = d$. Then

$$\left| \Omega \cap \Gamma_1(y) \right| = c_d + a_d$$

by Lemma 2.6(i).

Now assume (i)-(iii) hold. Then (4.23)-(4.24) hold by (4.21)-(4.22), and since Ω has valency $c_d + a_d$. This proves Theorem 4.6.

5. Distance-regular graphs with $c_2 > 1$.

In this section, we restrict our attention to the case $\Gamma = (X, R)$ is distance-regular with intersection number $c_2 > 1$. We first prove that a weak-geodetically closed subgraph Ω of Γ is regular (and consequently distance-regular by Theorem 4.6). We then give a precise description of Ω.

Lemma 5.1. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 2$, and assume the intersection number $c_2 > 1$. Fix a subgraph Ω of Γ, and pick any vertex $x \in \Omega$. Suppose Ω is weak-geodetically closed with respect to x. Then for all $y \in \Omega \cap \Gamma_1(x)$,

$$\Big|\Omega \cap \Gamma_1(x)\Big| \geq \Big|\Omega \cap \Gamma_1(y)\Big|.$$

Proof. Since Γ is regular, it suffices to prove

$$\Big|\Gamma_1(x) \setminus \Omega\Big| \leq \Big|\Gamma_1(y) \setminus \Omega\Big|. \tag{5.1}$$

Observe that each vertex in $\Gamma_1(x) \setminus \Omega$ is adjacent to $c_2 - 1$ vertices in $\Gamma_1(y) \setminus \Omega$. Indeed pick $z \in \Gamma_1(x) \setminus \Omega$. Then by Lemma 2.4(i),

$$\delta(z, y) = 2.$$

Note that z is adjacent to c_2 vertices in $\Gamma_1(y)$, and x is the unique one of such vertices in Ω by Lemma 2.4(ii). Hence z is adjacent to $c_2 - 1$ vertices in $\Gamma_1(y) \setminus \Omega$.

Next, observe that each vertex in $\Gamma_1(y) \setminus \Omega$ is adjacent to at most $c_2 - 1$ vertices in $\Gamma_1(x) \setminus \Omega$. Indeed pick $w \in \Gamma_1(y) \setminus \Omega$. Then by Lemma 2.3(iii),

$$\delta(x, w) = 2.$$

Since $y \in \Omega \cap \Gamma_1(x)$, w is adjacent to at most $c_2 - 1$ vertices in $\Gamma_1(x) \setminus \Omega$.

Now by counting edges between $\Gamma_1(x) \setminus \Omega$ and $\Gamma_1(y) \setminus \Omega$, we have

$$\Big|\Gamma_1(x) \setminus \Omega\Big|(c_2 - 1) \leq \Big|\Gamma_1(y) \setminus \Omega\Big|(c_2 - 1), \tag{5.2}$$

and (5.1) follows since $c_2 > 1$. This proves Lemma 5.1.

Lemma 5.2. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 2$, and assume the intersection number $c_2 > 1$. Let Ω denote a weak-geodetically closed subgraph of Γ. Then Ω is regular.

Proof. Suppose Ω is not regular. Since Ω is connected, there exist adjacent vertices $x, y \in \Omega$ such that

$$\Big|\Omega \cap \Gamma_1(x)\Big| < \Big|\Omega \cap \Gamma_1(y)\Big|, \tag{5.3}$$

contradicting Lemma 5.1.

Corollary 5.3. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 2$, and assume the intersection number $c_2 > 1$. Let Ω denote a weak-geodetically closed subgraph of Γ. Then Ω is distance-regular with intersection array

$$\{c_1, c_2, \cdots, c_d;\ b_0 - b_d, b_1 - b_d, \cdots, b_{d-1} - b_d\},$$

where $d = d(\Omega)$.

Proof. This is immediate from Lemma 5.2, Theorem 4.6 and (1.9).

Corollary 5.4. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 2$, and assume the intersection number $c_2 > 1$. Let Ω, Ω' denote two weak-geodetically closed subgraphs such that $\Omega' \subseteq \Omega$. Then the following (i)-(ii) are equivalent.

(i) $\Omega' = \Omega$.

(ii) $\mathrm{diam}(\Omega') = \mathrm{diam}(\Omega)$.

Proof. (i)$\longrightarrow$(ii). Clear.

(ii)$\longrightarrow$(i). Ω, Ω' are distance-regular with the same intersection array by Corollary 5.3. Now we have $\left|\Omega\right| = \left|\Omega'\right|$, so $\Omega = \Omega'$.

Definition 5.5. Let $\Gamma = (X, R)$ be a graph. For any vertex $x \in X$, and any subset $C \subseteq X$, define

$$[x, C] := \{v \in X | \text{there exists } y \in C, \text{such that } \ \delta(x, v) + \delta(v, y) = \delta(x, y)\}.$$

The following proposition gives us a description of a weak-geodetically closed subgraph of a distance-regular graph with $c_2 > 1$.

Proposition 5.6. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 2$, and assume the intersection number $c_2 > 1$. Pick any subgraph Ω of Γ, and fix an integer $d \quad (0 \leq d \leq D)$. Then the following (i)-(ii) are equivalent.

(i) Ω is weak-geodetically closed with diameter d.

(ii) There exists a vertex $x \in \Omega$ that satisfies the following (iia)-(iid).

- (iia) Ω is weak-geodetically closed with respect to x.
- (iib) $\left|\Omega \cap \Gamma_1(x)\right| = c_d + a_d$.
- (iic) $\Omega = [x, C]$ for some $C \subseteq \Gamma_d(x)$.
- (iid) For all $v \in \Omega$ and for all $z \in X$, if z is adjacent to two distinct vertices in $C(v, x)$, then $z \in \Omega$.

Proof. (i)$\longrightarrow$(ii). Let x denote any vertex in Ω. (iia) is immediate from Definition 3.1. (iib) is immediate from Corollary 5.3 and (1.9). Suppose (iic) fails. Then there exists a vertex $w \in \Omega$ such that $\delta(x,w) < d$ and $B(w,x) \cap \Omega = \emptyset$. This contradicts Corollary 5.3. Hence we have (iic). To prove (iid), suppose z is adjacent to distinct vertices $w, w' \in C(v,x)$. Then $w, w' \in \Omega$ by Lemma 2.3(ii), so $z \in \Omega$ by Lemma 2.4(ii).

(ii)$\longrightarrow$(i). First, we prove Ω is weak-geodetically closed. To do this, by parts (ii), (iii) of Theorem 4.6, it suffices to show that Ω is regular. We will show each vertex in Ω has valency $c_d + a_d$. Note that by Lemma 2.6(i), for all $w \in C$,

$$\left|\Omega \cap \Gamma_1(w)\right| = c_d + a_d. \tag{5.4}$$

Claim. For all integers $j \quad (1 \leq j \leq d)$, and for all pairs of adjacent vertices $u, v \in \Omega$ such that $u \in \Gamma_{j-1}(x)$ and $v \in \Gamma_j(x)$, we have

$$\left|\Omega \cap \Gamma_1(u)\right| \geq \left|\Omega \cap \Gamma_1(v)\right|. \tag{5.5}$$

Proof of Claim. Since Γ is regular, to prove (5.5), it suffices to prove

$$\left|\Gamma_1(u) \setminus \Omega\right| \leq \left|\Gamma_1(v) \setminus \Omega\right|. \tag{5.6}$$

We will count the edges between $\Gamma_1(u) \setminus \Omega$ and $\Gamma_1(v) \setminus \Omega$ in two ways to establish (5.6). On the one hand, we prove that each vertex in $\Gamma_1(u) \setminus \Omega$ is adjacent to exactly $c_2 - 1$ vertices in $\Gamma_1(v) \setminus \Omega$. Pick $z \in \Gamma_1(u) \setminus \Omega$. Note that $z \in \Gamma_j(x)$ by Lemma 2.3(iii). We now show $\delta(z,v) = 2$. Obviously $\delta(z,v) \leq 2$, since z, u, v is a path. Observe $z \neq v$ by construction. Also z, v are not adjacent, otherwise $z \in A(v,x) \subseteq \Omega$ by Lemma 2.3(ii), a contradiction. Hence $\delta(z,v) = 2$. Next we show

$$\Omega \cap C(z,v) = \{u\}. \tag{5.7}$$

To see this, pick $w \in \Omega \cap C(z,v)$ and suppose $w \neq u$. Note that $w \notin C(v,x)$, otherwise z is adjacent to $u, w \in C(v,x)$, putting $z \in \Omega$ by (iid), a contradiction. Note that $w \notin A(v,x)$, otherwise $w \in A(v,x)$ and $z \in A(w,x)$, putting $z \in \Omega$ by Lemma 2.3(ii), a contradiction. Hence $w \in B(v,x)$. Now $z \in C(w,x)$, putting $z \in \Omega$ by Lemma 2.3(ii), a contradiction. Hence $w = u$ and we have (5.7). Now observe that by (5.7),

$$\begin{aligned}\left|\Gamma_1(z) \cap (\Gamma_1(v) \setminus \Omega)\right| &= \left|C(z,v) \setminus \{u\}\right| \\ &= c_2 - 1.\end{aligned}$$

Hence z is adjacent to exactly $c_2 - 1$ vertices in $\Gamma_1(v) \setminus \Omega$.

On the other hand, we show that each vertex in $\Gamma_1(v) \setminus \Omega$ is adjacent to at most $c_2 - 1$ vertices in $\Gamma_1(u) \setminus \Omega$. Pick a vertex $z \in \Gamma_1(v) \setminus \Omega$. Observe $\delta(x, z) = j + 1$ by Lemma 2.3(iii). Observe $\delta(u, z) = 2$. Now we have the desired property, since z is adjacent to c_2 vertices in $\Gamma_1(u)$ and $v \in \Omega$ is one of them.

Using above the two ways to count the edges between vertices in $\Gamma_1(u) \setminus \Omega$ and vertices in $\Gamma_1(v) \setminus \Omega$, we have

$$\Big|\Gamma_1(u) \setminus \Omega\Big|(c_2 - 1) \le \Big|\Gamma_1(v) \setminus \Omega\Big|(c_2 - 1),$$

and (5.6) follows since $c_2 > 1$. This proves the claim.

To show Ω is regular, fix any geodesic path $x = x_0, x_1, \cdots, x_d$, where $x_d \in C$, and set

$$t_l := \Big|\Omega \cap \Gamma_1(x_l)\Big| \qquad (0 \le l \le d).$$

Observe

$$t_0 = a_d + c_d \tag{5.8}$$

by assumption (iib),

$$t_d = a_d + c_d \tag{5.9}$$

by (5.4), and

$$t_{l-1} \ge t_l \qquad (1 \le l \le d) \tag{5.10}$$

by the claim. It follows from (5.8)-(5.10) that

$$t_l = a_d + c_d \qquad (0 \le l \le d).$$

By Definition 5.5, Ω is the union of geodesic paths of the above type, and we conclude every vertex in Ω has valency $a_d + c_d$. Now Ω is weak-geodetically closed by Theorem 4.6. It remains to show Ω has diameter d. This holds, since Ω is distance-regular by Corollary 5.3 and $\mathrm{diam}_x(\Omega) = d$ by (iic). This proves Proposition 5.6.

If we assume $d = 2$ and $a_2 \ne 0$ in Proposition 5.6, we get the following improvement.

Proposition 5.7. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \ge 2$, and assume the intersection numbers $c_2 > 1$, $a_2 \ne 0$. Then for any subgraph Ω of Γ, the following (i)-(ii) are equivalent.

(i) Ω is weak-geodetically closed with diameter 2.

(ii) There exists a vertex $x \in \Omega$ that satisfies the following (iia)-(iic).

(iia) Ω is weak-geodetically closed with respect to x.

(iib) $\left|\Omega \cap \Gamma_1(x)\right| = a_2 + c_2$.

(iic) $\mathrm{diam}_x(\Omega) = 2$.

Proof. (i)⟶(ii). (iia)-(iic) are immediate from Proposition 5.6 (iia)-(iic) (with $d = 2$).

(ii)⟶(i). First, we prove Ω is weak-geodetically closed. To do this, in view of parts (ii), (iii) of Theorem 4.6, it suffices to show Ω is regular. We show each vertex in Ω has valency $c_2 + a_2$. Observe by Lemma 2.6(i) that for all $y \in \Omega \cap \Gamma_2(x)$,

$$\left|\Omega \cap \Gamma_1(y)\right| = c_2 + a_2. \tag{5.11}$$

Observe by Lemma 5.1 that for all $z \in \Omega \cap \Gamma_1(x)$,

$$\left|\Omega \cap \Gamma_1(z)\right| \leq c_2 + a_2. \tag{5.12}$$

It remains to show equality holds in (5.12) for all $z \in \Omega \cap \Gamma_1(x)$. We suppose this is not the case and get a contradiction. Pick $z' \in \Omega \cap \Gamma_1(x)$ such that $\left|\Omega \cap \Gamma_1(z')\right|$ is minimal and assume

$$\left|\Omega \cap \Gamma_1(z')\right| < c_2 + a_2. \tag{5.13}$$

Claim 1. There exists a vertex $y \in \Omega \cap \Gamma_2(x)$ that is not adjacent to z'.

Proof of Claim 1. If this fails, then z' is adjacent to each vertex in $\Omega \cap \Gamma_2(x)$. Hence by Lemma 2.3(ii) and the construction, for all $z \in \Omega \cap \Gamma_1(x)$,

$$\begin{aligned}\left|\Omega \cap B(z,x)\right| &= \left|\Omega \cap \Gamma_1(z)\right| - c_1 - a_1\\ &\geq \left|\Omega \cap \Gamma_1(z')\right| - c_1 - a_1\\ &= \left|\Omega \cap \Gamma_2(x)\right|.\end{aligned}$$

Then every vertex in $\Omega \cap \Gamma_1(x)$ is adjacent to every vertex in $\Omega \cap \Gamma_2(x)$. But this is inconsistent with (iib) and $a_2 \neq 0$. Hence we have Claim 1.

We fix y, z' for the rest of this proof. Observe, by (iib), Lemma 2.6(iv),

$$C(x,y) \cup A(x,y) = \Omega \cap \Gamma_1(x).$$

Now set

$$\gamma := \frac{1}{c_2} \sum_{z \in C(x,y)} \left|\Omega \cap \Gamma_1(z)\right|, \tag{5.14}$$

and observe γ is the average valency (in Ω) of a vertex in $C(x,y)$. Similarly, set

$$\lambda := \frac{1}{a_2} \sum_{z \in A(x,y)} \left| \Omega \cap \Gamma_1(z) \right|, \tag{5.15}$$

and observe λ is the average valency (in Ω) of a vertex in $A(x,y)$.

Claim 2. $\lambda < a_2 + c_2$.

Proof of Claim 2. This is immediate from (5.12), (5.13), (5.15), and the observation $z' \in A(x,y)$.

Now set

$$\Delta := \{w \in \Omega | \delta(x,w) = 2, \delta(y,w) \geq 2\}.$$

Claim 3.

$$\left|\Delta\right| c_2 + a_2 c_2 + c_2 = c_2(\gamma - a_1 - 1) + a_2(\lambda - a_1 - 1). \tag{5.16}$$

Proof of Claim 3. Let e denote the number of edges connecting vertices in $\Omega \cap \Gamma_1(x)$ to vertices in $\Omega \cap \Gamma_2(x)$. We count e in two ways. On the one hand,

$$\begin{aligned} e &= \left|\Omega \cap \Gamma_2(x)\right| c_2 \\ &= \left|\Delta \cup A(y,x) \cup \{y\}\right| c_2 \\ &= (\left|\Delta\right| + a_2 + 1) c_2. \end{aligned} \tag{5.17}$$

On the other hand,

$$\begin{aligned} e &= \left|C(x,y)\right| (\gamma - a_1 - 1) + \left|A(x,y)\right| (\lambda - a_1 - 1) \\ &= c_2(\gamma - a_1 - 1) + a_2(\lambda - a_1 - 1). \end{aligned} \tag{5.18}$$

Line (5.16) is immediate from (5.17), (5.18), and Claim 3 is proved.

Claim 4.

$$\left|\Delta\right| c_2 + a_2 c_2 + c_2 \geq c_2(\gamma - a_1 - 1) + a_2(a_2 + c_2 - a_1 - 1). \tag{5.19}$$

Proof of Claim 4. Let f denote the number of edges connecting vertices in $\Omega \cap \Gamma_1(y)$ to vertices in $\Omega \cap \Gamma_2(y)$. Again, we count f in two ways. On the one hand,

$$\begin{aligned} f &\leq \left|\Omega \cap \Gamma_2(y)\right| c_2 \\ &\leq \left|\Delta \cup A(x,y) \cup \{x\}\right| c_2 \\ &= \left|\Delta\right| c_2 + a_2 c_2 + c_2, \end{aligned} \tag{5.20}$$

and on the other hand, using (5.11),

$$
\begin{aligned}
f &\geq \left|C(y,x)\right|(\gamma - a_1 - 1) + \left|A(y,x)\right|(a_2 + c_2 - a_1 - 1) \\
&= c_2(\gamma - a_1 - 1) + a_2(a_2 + c_2 - a_1 - 1). \qquad (5.21)
\end{aligned}
$$

(5.19) is immediate from (5.20), (5.21), and Claim 4 is proved.

Now subtracting (5.16) from (5.19), we find

$$0 \geq a_2(a_2 + c_2 - \lambda).$$

But this is impossible since $a_2 > 0$ by assumption, and $a_2 + c_2 - \lambda > 0$ by Claim 2. Hence equality holds in (5.12) for all $z \in \Omega \cap \Gamma_1(x)$. Now Ω is regular by (iib), (5.11), so Ω is weak-geodetically closed by Theorem 4.6. It remains to show Ω has diameter 2. This holds, since Ω is distance-regular by Corollary 5.3, and since $\mathrm{diam}_x(\Omega) = 2$ by (iic). This proves Proposition 5.7.

6. Distance-regular graphs with many weak-geodetically closed subgraphs.

In this section, we obtain our first major result, Theorem 6.4. To describe it, we need a few definitions.

Definition 6.1. Let $\Gamma = (X, R)$ be a graph with diameter D, and let i denote an integer $(0 \leq i \leq D)$. Then Γ is said to be $i-$*bounded*, if for all integers j $(0 \leq j \leq i)$, and for all x, $y \in X$ such that $\delta(x, y) = j$, x, y are contained in a common weak-geodetically closed subgraph of diameter j.

Lemma 6.2. Let $\Gamma = (X, R)$ be a graph with diameter $D \geq 1$. Then the following (i)-(iii) hold.

(i) Γ is 0-bounded.

(ii) For each integer i $(1 \leq i \leq D)$, if Γ is i-bounded then Γ is $(i-1)$-bounded.

(iii) Suppose Γ is $(D-1)$-bounded. Then Γ is D-bounded.

Proof. (i)-(iii) are clear from Definition 6.1.

In Theorem 6.4, we obtain a simple criterion for a distance-regular graph Γ to be i-bounded. We will use the following notation.

Definition 6.3. Let $\Gamma = (X, R)$ be a graph with diameter $D \geq 2$. Pick an integer i $(2 \leq i \leq D)$. By a *parallelogram* of length i in Γ, we mean a 4-tuple $xyzw$ of vertices of X such that

$$\delta(x, y) = \delta(z, w) = 1, \quad \delta(x, w) = i,$$

$$\delta(x,z)=\delta(y,z)=\delta(y,w)=i-1.$$

We now state the first main theorem of our paper.

Theorem 6.4. Let $\Gamma=(X,R)$ be a distance-regular graph with diameter $D\geq 2$, and assume the intersection numbers $c_2>1$, $a_1\neq 0$. Pick an integer i $(1\leq i<D)$. Then the following (i)-(ii) are equivalent.

(i) Γ is i-bounded.

(ii) Γ contains no parallelogram of length $\leq i+1$.

The following Lemma proves Theorem 6.4(i)$\longrightarrow$(ii).

Lemma 6.5. Let $\Gamma=(X,R)$ be a distance-regular graph with diameter $D\geq 2$. Pick an integer i $(1\leq i<D)$. Suppose Γ is i-bounded. Then Γ contains no parallelogram of length $\leq i+1$.

Proof. Suppose Γ contains a parallelogram $xyzw$ of some length $j\leq i+1$. Then $\delta(y,z)=j-1\leq i$. By Definition 6.1, there exists a weak-geodetically closed subgraph Ω of Γ that has diameter $j-1$ and contains y,z. Observe $x\in A(y,z)$ and $w\in A(z,y)$, so $x,w\in\Omega$ by Lemma 2.3(ii). But $\delta(x,w)=j$, contradicting our assumption that Ω has diameter $j-1$. This proves the lemma.

We prove Theorem 6.4 (ii)$\longrightarrow$(i) by induction on i. We deal with the case $i=1$ in Lemma 6.6, the case $i=2$ in Proposition 6.7, prove some general results in Proposition 6.8-Lemma 6.13, and then proceed to the case $i\geq 3$ at the end of this section.

Lemma 6.6. Let $\Gamma=(X,R)$ be a distance-regular graph with diameter $D\geq 2$. Suppose that Γ contains no parallelogram of length 2. Then Γ is 1-bounded.

Proof. Pick $x,y\in X$ with $\delta(x,y)=1$, and set $\Omega:=A(x,y)\cup\{x,y\}$. Ω is a clique of size a_1+2 since Γ contains no parallelograms of length 2; in particular, Ω has diameter 1. Also Ω is weak-geodetically closed by Theorem 4.6, since Ω is regular and equality holds in (4.12). This proves Lemma 6.6

Our proof of the case $i=2$ in Theorem 6.4 (ii)$\longrightarrow$(i) is different from our proof for the case $i\geq 3$, also, we can prove it under the assumption $a_2\neq 0$ instead of $a_1\neq 0$(One can easily show if Γ contains no parallelogram of length 2 then $a_1\neq 0$ implies $a_2\neq 0$). Hence we prove it separately.

Proposition 6.7. Let $\Gamma=(X,R)$ be a distance-regular graph with diameter $D\geq 2$. Assume that the intersection numbers $c_2>1$, $a_2\neq 0$. Suppose that Γ contains no parallelogram of length ≤ 3. Then Γ is 2-bounded.

Proof. Pick any vertices $x,y\in X$ with $\delta(x,y)=2$. Let C be the connected component of $\Gamma_2(x)$ containing y. Set $\Omega:=[x,C]$ as in Definition 5.5. We prove

Ω is weak-geodetically closed of diameter 2. To do this, we show Ω satisfies (iia)-(iic) of Proposition 5.7.

Claim 1. Ω is weak-geodetically closed with respect to x. In particular, (iia) of Proposition 5.7 is satisfied.

Proof of Claim 1. Fix $z \in \Omega$. By Lemma 2.3(ii) and the construction, it suffices to show $A(z, x) \subseteq \Omega$. This clearly holds if $z = x$ or $z \in \Omega \cap \Gamma_2(x)$, so assume $z \in \Omega \cap \Gamma_1(x)$. Pick $w \in A(z, x)$. By construction, there exists $z' \in C$ such that $z \in C(z', x)$. Observe that $\delta(w, z') = 2$; otherwise $\delta(w, z') = 1$ and $xzwz'$ is a parallelogram of length 2, a contradiction. Pick $w' \in C(w, z') \setminus \{z\}$. Observe that $w' \in C(z', x) \cup A(z', x)$. Suppose $w' \in C(z', x)$. Then $\delta(z, w') = 2$; otherwise $\delta(z, w') = 1$ and $z'zw'x$ is a parallelogram of length 2, a contradiction. But now $w'wxz$ is a parallelogram of length 2, a contradiction. Hence $w' \in A(z', x)$, forcing $w' \in C$ by construction. Now $w \in \Omega$ by construction. This proves Claim 1.

Claim 2. For all adjacent vertices $z, z' \in C$, $B(x, z) = B(x, z')$. In particular, $B(x, w) = B(x, w')$ for all $w, w' \in C$.

Proof of Claim 2. Fix adjacent vertices $z, z' \in C$. By symmetry, it suffices to prove $B(x, z) \subseteq B(x, z')$. Suppose there exists a vertex $p \in B(x, z) \setminus B(x, z')$. Of course $\delta(x, p) = 1$, $\delta(z, p) = 3$ by construction, so $\delta(z', p) = 2$ by the triangular inequality. Now the 4-tuple $pxz'z$ is a parallelogram of length 3, a contradiction. Hence $B(x, z) = B(x, z')$. Since C is connected, we have $B(x, w) = B(x, w')$ for all $w, w' \in C$. This proves Claim 2.

Claim 3. $\Big|\Omega \cap \Gamma_1(x)\Big| = c_2 + a_2$. In particular, (iib) of Proposition 5.7 is satisfied.

Proof of Claim 3. Pick $z \in C$. Then it suffices to show $\Omega \cap \Gamma_1(x) = C(x, z) \cup A(x, z)$. By Claim 1, Ω is weak-geodetically closed with respect to x. Hence by Lemma 2.6(ii), $C(x, z) \cup A(x, z) \subseteq \Omega \cap \Gamma_1(x)$. Since $\Gamma_1(x) = C(x, z) \cup A(x, z) \cup B(x, z)$, it remains to show $\Omega \cap B(x, z) = \emptyset$. Suppose there exists $w \in \Omega \cap B(x, z)$. By construction, there exists $w' \in C$ such that $w \in C(x, w')$. But $w \in B(x, z) = B(x, w')$ by Claim 2, a contradiction. Hence $\Omega \cap B(x, z) = \emptyset$, as desired. This proves Claim 3.

Note that (iic) of Proposition 5.7 is satisfied by the construction. Hence Ω is weak-geodetically closed with diameter 2 by Proposition 5.7. We now have Proposition 6.7.

Proposition 6.8. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 2$. Assume the intersection numbers $c_2 > 1$, $a_2 \neq 0$. Suppose Γ contains no parallelogram of length 2 and suppose there exists a weak-geodetically closed subgraph Ω of diameter 2. Fix a vertex $x \in \Omega$. Then $\Omega \cap \Gamma_2(x)$ is connected.

Proof. Note that Ω is distance-regular by Corollary 5.3. Suppose that $\Omega \cap \Gamma_2(x)$ is not connected. Pick $u, v \in \Omega \cap \Gamma_2(x)$ such that there is no path in $\Omega \cap \Gamma_2(x)$ connecting u, v. Observe $\delta(u, v) = 2$, since Ω has diameter 2. For each vertex $z \in C(u, v)$, we must have $z \in C(u, x)$, otherwise $\delta(x, z) = 2$ and u, z, v is a path in $\Omega \cap \Gamma_2(x)$. Hence we have $C(u, v) \subseteq C(u, x)$. Now $C(u, v) = C(u, x)$, since both sets have the same cardinality c_2. Similarly, we have $C(u, v) = C(v, x)$. Pick $w \in A(u, v)$. Observe $\delta(x, w) = 2$, since $w \notin C(u, v) = C(u, x)$. We do not have a path in $\Omega \cap \Gamma_2(x)$ connecting w, v, otherwise we can extend this path to a path in $\Omega \cap \Gamma_2(x)$ connecting u, v. By the same argument as above, we have $C(w, v) = C(w, x) = C(v, x)$. Now we have

$$\begin{aligned} C(u, v) &= C(v, x) \\ &= C(w, v). \end{aligned}$$

Pick distinct vertices $z, z' \in C(u, v) = C(w, v)$. If $\delta(z, z') = 1$ then the 4-tuple $uzz'v$ is a parallelogram of length 2, a contradiction. If $\delta(z, z') = 2$ then the 4-tuple $zuwz'$ is a parallelogram of length 2, another contradiction. Hence we prove $\Omega \cap \Gamma_2(x)$ is connected.

Note. Proposition 6.8 tells us that in the case $d = 2$ of Proposition 5.6(iic), some $C \subseteq \Gamma_d(x)$ is connected. We do not know if this is true in general situation $d > 2$.

Lemma 6.9. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$. Suppose the intersection numbers $c_2 > 1$, $a_2 \neq 0$. Pick an integer i $(2 \leq i < D)$, and suppose Γ contains no parallelogram of any length $\leq i + 1$. Let x be a vertex of Γ, and let Ω be a weak-geodetically closed subgraph of Γ with diameter 2. Suppose there exists a vertex $u \in \Omega \cap \Gamma_{i-1}(x)$, and suppose $\Omega \cap \Gamma_{i+1}(x) \neq \emptyset$. Then for all $t \in \Omega$, we have $\delta(x, t) = i - 1 + \delta(u, t)$.

Proof. We prove this by induction on the integer i. The case $i = 2$ is immediate from Lemma 2.4(i), so suppose $i > 2$. Note that

$$\Omega \subseteq \Gamma_{i-1}(x) \cup \Gamma_i(x) \cup \Gamma_{i+1}(x),$$

since $\mathrm{diam}(\Omega) = 2$, $\Omega \cap \Gamma_{i-1}(x) \neq \emptyset$, and $\Omega \cap \Gamma_{i+1}(x) \neq \emptyset$. We need to prove $\Omega \cap \Gamma_1(u) \subseteq \Gamma_i(x)$ and $\Omega \cap \Gamma_2(u) \subseteq \Gamma_{i+1}(x)$. It suffices to prove that $\Omega \cap \Gamma_2(u) \subseteq \Gamma_{i+1}(x)$, since Ω is distance-regular with diameter 2 and for each vertex $w \in \Omega \cap \Gamma_1(u)$, $w \in C(z, u)$ for some vertex $z \in \Omega \cap \Gamma_2(u)$. Suppose that $\Omega \cap \Gamma_2(u) \not\subseteq \Gamma_{i+1}(x)$. Since $\Omega \cap \Gamma_2(u)$ is connected by Proposition 6.8, and since $\Omega \cap \Gamma_2(u) \cap \Gamma_{i+1}(x) = \Omega \cap \Gamma_{i+1}(x) \neq \emptyset$, there exist adjacent vertices $v, v' \in \Gamma_2(u) \cap \Omega$ such that $v \in \Gamma_{i+1}(x)$ and $v' \in \Gamma_i(x)$. Pick $x' \in C(x, u)$. Then $\delta(x', u) = i - 2$ and $\delta(x', v) = i$. By induction hypothesis, we have $\delta(x', v') = i$. Now the 4-tuple $xx'v'v$ is a parallelogram of length $i + 1$, a contradiction. This proves the lemma.

Corollary 6.10. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$. Assume the intersection numbers $c_2 > 1$, $a_2 \neq 0$. Pick any integer i $(2 \leq i < D)$, and suppose Γ contains no parallelogram of any length $\leq i+1$. Let x be a vertex of Γ, and let Ω be a weak-geodetically closed subgraph of diameter 2. If there exist 2 distinct vertices u, v in Ω such that $\delta(x,u) = \delta(x,v) = i-1$, then $\delta(x,t) \leq i$ for all vertices $t \in \Omega$.

Proof. Suppose this is false. Then $\Omega \cap \Gamma_{i+1}(x) \neq \emptyset$, so $\delta(x,v) = \delta(x,u) + \delta(u,v)$ by Lemma 6.9. Since $\delta(x,v) = \delta(x,u) = i-1$, we have $\delta(u,v) = 0$ and hence $u = v$, a contradiction. This proves the corollary.

Definition 6.11. Let $\Gamma = (X, R)$ be a graph with diameter $D \geq 2$. Pick an integer i $(2 \leq i \leq D)$. By a *kite* of length i, we mean a 4-tuple $xyzw$ of vertices of Γ such that $\{x, y, z\}$ is a clique, and w is at distances

$$\delta(w,x) = i, \quad \delta(w,y) = i-1, \quad \delta(w,z) = i-1.$$

Note. A kite of length 2 is the same thing as a parallelogram of length 2.

Lemma 6.12. Let $\Gamma = (X, R)$ be a graph with diameter $D \geq 2$. Fix an integer i $(2 \leq i \leq D)$. Suppose Γ contains no parallelogram of any length $\leq i$. Then Γ contains no kite of any length $\leq i$.

Proof. Suppose Γ contains a kite of length $\leq i$. Of all these kites, pick a kite $xyzw$ with minimal length j. Observe $j \neq 2$, otherwise $xyzw$ is a parallelogram of length 2. Now pick $a \in C(w, z)$. Note that $\delta(a, z) = j - 2$. Observe

$$\begin{aligned} \delta(a,y) &\leq \delta(a,z) + \delta(z,y) \\ &= j - 2 + 1 \\ &= j - 1, \end{aligned}$$

and

$$\begin{aligned} \delta(a,y) &\geq \delta(y,w) - \delta(a,w) \\ &= j - 1 - 1 \\ &= j - 2, \end{aligned}$$

so $\delta(a,y) = j-2$ or $\delta(a,y) = j-1$. If $\delta(a,y) = j-2$, then the 4-tuple $xyza$ is a kite of length $j-1$, contradicting our construction, so $\delta(a,y) = j-1$. Now the 4-tuple $wayx$ is a parallelogram of length j, a contradiction. This proves the lemma.

Lemma 6.13. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D \geq 3$. Assume the intersection numbers $c_2 > 1$ and $a_1 \neq 0$. Pick an integer i

$(2 \le i < D)$, and suppose Γ contains no parallelogram of any length $\le i+1$. Let x be a vertex of X, and let Ω be a weak-geodetically closed subgraph of diameter 2. Set $j :=\max\{\delta(x,w) \mid w \in \Omega\}$, and assume $j \le i$. Then $\Omega \cap \Gamma_j(x)$ is connected.

Proof. Note that Ω is distance-regular by Corollary 5.3. Since Γ contains no parallelogram of length 2, for any vertex $u \in \Omega$, $\Omega \cap \Gamma_1(u)$ is a disjoined union of cliques of size a_1+1. Let l denote the number of these cliques in $\Omega \cap \Gamma_1(u)$. Suppose $\Omega \cap \Gamma_j(x)$ is not connected. Then there exist two vertices $t, s \in \Omega \cap \Gamma_j(x)$ such that there is no path in $\Omega \cap \Gamma_j(x)$ connecting t and s. Note that $\delta(t,s)=2$, since $\mathrm{diam}(\Omega)=2$. Consider the set

$$N := \{t\} \cup (\Gamma_j(x) \cap \Omega \cap \Gamma_1(t)).$$

Claim 1. $\Big|N\Big| \ge 1 + la_1$.

Proof of Claim 1. Γ contains no kite of length j by Lemma 6.12, , so $\Big|K \cap \Gamma_{j-1}(x)\Big| \le 1$ for all maximal cliques $K \subseteq \Omega \cap \Gamma_1(t)$. Hence $\Big|K \cap N\Big| \ge a_1$ for all maximal cliques $K \subseteq \Omega \cap \Gamma_1(t)$. Since there are l such cliques,

$$\begin{aligned}\Big|N\Big| &\ge \Big|\{x\}\Big| + la_1 \\ &= 1 + la_1,\end{aligned}$$

as desired.

Claim 2. $\delta(z,s)=2$ for any $z \in N$.

Proof of Claim 2. $\delta(z,s) \le 2$, since $\mathrm{diam}(\Omega)=2$. $z \ne s$ by construction. Also $\delta(z,s) \ne 1$, otherwise t,z,s is a path in $\Omega \cap \Gamma_j(x)$, a contradiction. Hence $\delta(z,s)=2$.

Now consider the set

$$M := \bigcup_{z \in N} C(s,z).$$

Claim 3. $\Big|M\Big| \le l$.

Proof of Claim 3. Note that $M \subseteq \Omega \cap \Gamma_1(s)$ by construction and Lemma 2.3(ii), so there is no vertex in M with distance $j+1$ to x. If the distance from x to a vertex in M is j, then we find a path in $\Omega \cap \Gamma_j(x)$ connecting t and s, a contradiction. Thus

$$M \subseteq \Omega \cap \Gamma_1(s) \cap \Gamma_{j-1}(x).$$

Since $\delta(x,s)=j$ and since Γ contains no kite of length j by Lemma 6.12, each maximal clique in $\Omega \cap \Gamma_1(s)$ contains at most one vertex in M. Hence $|M| \le l$.

We count the number e of edges between vertices in N and vertices in M in two ways. On the one hand, by Claim 1, we have

$$\begin{aligned} e &= \left|N\right| c_2 \\ &\geq (1 + la_1)c_2. \end{aligned} \tag{6.1}$$

On the other hand, a vertex in $M \cap \Gamma_1(t)$ is a adjacent to at most $a_1 + 1$ vertices in N, and a vertex in $M \cap \Gamma_2(t)$ is adjacent to at most c_2 vertices in N. Observe that $a_1c_2 \geq \max\{a_1 + 1, c_2\}$, since $c_2 > 1$ and $a_1 \neq 0$. Hence each vertex in M is adjacent to at most a_1c_2 vertices in N. Now by Claim 3,

$$\begin{aligned} e &\leq \left|M\right| a_1c_2 \\ &\leq la_1c_2, \end{aligned}$$

a contradiction to (6.1). This proves the lemma.

Note. Lemma 6.13 is the only place we need the assumption $a_1 \neq 0$ instead of the assumption $a_2 \neq 0$.

Proof of Theorem 6.4 (ii)⟶(i). This is by induction on the integer i. The cases $i = 1$ and $i = 2$ hold by Lemma 6.6 and Proposition 6.7, so assume $i \geq 3$. Fix vertices $x, y \in X$ with $\delta(x, y) = i$. Let C be the connected component in $\Gamma_i(x)$ containing y. Let $\Omega := [x, C]$ be as in Definition 5.5. We prove Ω is weak-geodetically closed of diameter i. To do this, we show Ω satisfies (iia)-(iid) of Proposition 5.6.

Claim 1. Let z, z', w be vertices in X, with $\delta(x, w) \leq i$, $z \in C(w, x)$, $z' \in A(z, x)$. Then $\delta(w, z') = 2$. Moreover, there exists a vertex $w' \in C(z', w) \cap A(w, x)$.

Proof of Claim 1. By Lemma 6.12, Γ contains no kite of length $\leq i$. Now $\delta(w, z') = 2$, otherwise $wzz'x$ is a kite of length $\delta(x, w)$. Since $c_2 > 1$, there exists $w' \in C(z', w) \setminus \{z\}$. Note that either $w' \in A(z', x)$ or $w' \in A(w, x)$. Suppose $w' \in A(z', x)$. By the induction hypothesis, x, z' are contained in a common weak-geodetically closed subgraph Ω' of diameter $\delta(x, z')$. Observe that by Lemma 2.3(ii), $w', z \in \Omega'$ and by Lemma 2.4(ii), $w \in \Omega'$. But

$$\begin{aligned} \delta(x, w) &= \delta(x, z) + 1 \\ &= \delta(x, z') + 1 \\ &> \delta(x, z'), \end{aligned}$$

a contradiction. Hence $w' \in A(w, x)$.

Claim 2. Ω is weak-geodetically closed with respect to x. In particular, (iia) of Proposition 5.6 is satisfied.

Proof of Claim 2. By Lemma 2.3(ii), it suffices to check $C(z,x) \cup A(z,x) \subseteq \Omega$ for all $z \in \Omega$. If this is not the case, then there is a vertex $z \in \Omega$ such that $C(z,x) \cup A(z,x) \nsubseteq \Omega$. Of all such vertices z, we pick one with $\delta(x,z)$ maximum. Observe $\delta(x,z) < i$ by construction, so there is a vertex $w \in \Omega$ such that $z \in C(w,x)$. Pick $z' \in C(z,x) \cup A(z,x) \setminus \Omega$. Note that $C(z,x) \subseteq \Omega$ by Definition 5.5, so $z' \in A(z,x)$. By Claim 1, there is a vertex $w' \in C(z',w) \cap A(w,x)$. By the choice of z and since $\delta(x,w) > \delta(x,z)$, we have $w' \in A(w,x) \subseteq \Omega$. But now $z' \in C(w',x) \subseteq \Omega$, a contradiction.

Claim 3. For all adjacent vertices $z, z' \in C$, $B(x,z) = B(x,z')$. In particular, $B(x,w) = B(x,w')$ for all $w, w' \in C$.

Proof of Claim 3. Fix adjacent vertices $z, z' \in C \subseteq \Gamma_i(x)$. By symmetry, it suffices to prove $B(x,z) \subseteq B(x,z')$. Suppose there exists a vertex $p \in B(x,z) \setminus B(x,z')$. Of course $\delta(x,p) = 1$, $\delta(z,p) = i+1$ by construction, so $\delta(z',p) = i$ by the triangular inequality. Now the 4-tuple $pxz'z$ is a parallelogram of length $i+1$, a contradiction. Hence $B(x,z) = B(x,z')$. Since C is connected, we have $B(x,w) = B(x,w')$ for all $w, w' \in C$. This proves Claim 3.

Claim 4. $\Big|\Omega \cap \Gamma_1(x)\Big| = c_i + a_i$. In particular, (iib) of Proposition 5.6 is satisfied.

Proof of Claim 4. Pick $z \in C$. Then it suffices to show $\Omega \cap \Gamma_1(x) = C(x,z) \cup A(x,z)$. By Claim 2, Ω is weak-geodetically closed with respect to x. Hence by Lemma 2.6(ii), $C(x,z) \cup A(x,z) \subseteq \Omega \cap \Gamma_1(x)$. Since $\Gamma_1(x) = C(x,z) \cup A(x,z) \cup B(x,z)$, it remains to show $\Omega \cap B(x,z) = \emptyset$. Suppose there exists $w \in \Omega \cap B(x,z)$. By construction, there exists $w' \in C$ such that $w \in C(x,w')$. But $w \in B(x,z) = B(x,w')$ by Claim 3, a contradiction. Hence $\Omega \cap B(x,z) = \emptyset$, as desired. This proves Claim 4.

Claim 5. (iid) of Proposition 5.6 is satisfied.

Proof of Claim 5. Fix $v \in \Omega$ and $z \in X$ such that z is adjacent to two distinct vertices $u, w \in C(v,x)$. We need to prove $z \in \Omega$. Set $j := \delta(x,v)$. Note $j \leq i$ and $\delta(x,u) = \delta(x,w) = j-1$. Observe that $\delta(z,v) \leq 2$ and $u, w \in \Omega$ by construction. We can assume $z \in B(u,x)$; otherwise $z \in A(u,x) \cup C(u,x) \subseteq \Omega$ by Claim 2, Lemma 2.3(ii), and we are done. Now $\delta(x,z) = j$. We also can assume $\delta(z,v) = 2$, otherwise $v = z \in \Omega$ or $z \in A(v,x) \subseteq \Omega$ by Claim 2, Lemma 2.3(ii), and we are done. By the induction hypothesis, Γ is $(i-1)$-bounded. Especially, since $i \geq 3$, Γ is 2-bounded. Let Ω' be the weak-geodetically closed subgraph of Γ that has diameter 2 and contains z, v. Observe that $u, w \in C(z,v) \subseteq \Omega'$, and $a_2 \neq 0$ by Proposition 3.2(ii). Hence by Corollary 6.10, we have $\max\{\delta(x,s) \mid s \in \Omega'\} = j$. Now $\Omega' \cap \Gamma_j(x)$ is connected by Lemma 6.13. In particular, there is a path in $\Gamma_j(x)$ connecting v, z. Now by Claim 2 and Lemma 2.3(ii), we see each vertex in this path is in Ω, in particular, $z \in \Omega$. This proves Claim 5.

Of course, (iic) of Proposition 5.6 is satisfied. Hence Ω is weak-geodetically closed with diameter i by Proposition 5.6. We complete the proof of Theorem 6.4.

Problem. Can one prove Theorem 6.4(ii)$\longrightarrow$(i) under the assumption $a_2 \neq 0$ instead of the assumption $a_1 \neq 0$?

7. Distance-regular graphs with the Q-polynomial property.

In this section, we consider distance-regular graphs with the Q-polynomial property. Theorem 7.2 is our main result. First, we recall the definition of the Q-polynomial property. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$ and intersection numbers $p^h_{ij} \quad (0 \leq h, i, j \leq D)$. For each integer $i \quad (0 \leq i \leq D)$, the ith *distance matrix* A_i of Γ has rows and columns indexed by X, and x, y entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \delta(x,y) = i, \\ 0, & \text{if } \delta(x,y) \neq i \end{cases} \qquad (x, y \in X).$$

Then

$$A_0 = I, \tag{7.1}$$
$$A_i^t = A_i \qquad (0 \leq i \leq D), \tag{7.2}$$

and

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \qquad (0 \leq i, j \leq D) \tag{7.3}$$

[2, p127]. By (7.1)-(7.3), the matrices $A_0, A_1, \cdots, A_D$ form a basis for a commutative semi-simple real algebra M, known as the *Bose-Mesner algebra.* By [1,p59,p64], M has a second basis $E_0, E_1, \cdots, E_D$ such that

$$E_0 = \mid X \mid^{-1} J \qquad (J = \text{all } 1'\text{s matrix}), \tag{7.4}$$
$$E_i E_j = \delta_{ij} E_i \qquad (0 \leq i, j \leq D), \tag{7.5}$$
$$E_0 + E_1 + \cdots + E_D = I, \tag{7.6}$$
$$E_i^t = E_i \qquad (0 \leq i \leq D). \tag{7.7}$$

The $E_0, E_1, \cdots, E_D$ are known as the *primitive idempotents* of Γ, and E_0 is known as the *trivial* idempotent.

Let $\circ$ denote entry-wise multiplication of matrices. Then

$$A_i \circ A_j = \delta_{ij} A_i \qquad (0 \leq i, j \leq D),$$

so M is closed under $\circ$. Thus there exists real numbers $q_{ij}^h \quad (0 \leq i, j, h \leq D)$ such that

$$E_i \circ E_j = \mid X \mid^{-1} \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D).$$

Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \cdots, E_D$ of the primitive idempotents) if for all integers $h, i, j \quad (0 \leq h, i, j \leq D)$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two. Let E denote any nontrivial primitive idempotent of Γ. Then Γ is said to be *Q-polynomial with respect to* E whenever there exists an ordering $E_0, E_1 = E, E_2, \cdots, E_D$ of the primitive idempotents of Γ, with respect to which Γ is Q-polynomial.

The following is a special kind of Q-polynomial distance-regular graph[2, p193].

Definition 7.1. A distance-regular graph Γ is said to have *classical parameters* $(D,\ b,\ \alpha,\ \beta)$ whenever the diameter of Γ is D, and the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} (1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}) \qquad (0 \leq i \leq D), \tag{7.8}$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \qquad (0 \leq i \leq D), \tag{7.9}$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{j-1}. \tag{7.10}$$

Theorem 7.2. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$ and intersection numbers $c_2 > 1$, $a_1 \neq 0$. Assume Γ is Q-polynomial. Then the following (i)-(viii) are equivalent.

(i) Γ contains no parallelogram of any length.

(ii) Γ contains no parallelogram of length 2 or 3.

(iii) Γ contains no kite of any length.

(iv) Γ contains no kite of length 2 or 3.

(v) Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$, and either $b < -1$ or Γ is a dual polar graph or a Hamming graph.

(vi) Γ has classical parameters and contains no kite of length 2.

(vii) Γ is D-bounded.

(viii) Γ is 2-bounded.

(See [1,III.2] and [1, III.6] for definition of Hamming graphs and dual polar graphs).

We now mention a few items of notation, then prove a lemma, and then proceed to the proof of Theorem 7.2.

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Suppose Γ is Q-polynomial with respect to E. Then the *dual eigenvalues* $\theta_i^* \quad (0 \leq i \leq D)$ are defined by

$$E =\mid X \mid^{-1} \sum_{i=0}^{D} \theta_i^* A_i. \tag{7.11}$$

By [6, p384], the dual eigenvalues $\theta_i^* \quad (0 \leq i \leq D)$ are mutually distinct real numbers.

Set $V = \mathbb{R}^{|X|}$ (column vectors), and view the coordinates of V as being indexed by X. For each vertex $x \in X$, set

$$\hat{x} = (0, 0, \cdots, 1, 0, \cdots, 0)^t, \tag{7.12}$$

where 1 is in coordinate x. Also, let $\langle \ , \ \rangle$ denote the dot product

$$\langle u, v \rangle = u^t v \qquad (u, v \in V). \tag{7.13}$$

Then referring to the primitive idempotent E in (7.11), we compute from (7.7), (7.11)-(7.13) that for all $x, y \in X$,

$$\langle E\hat{x}, \hat{y} \rangle =\mid X \mid^{-1} \theta_i^*, \tag{7.14}$$

where $i = \delta(x, y)$.

Lemma 7.3. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$, and pick any integer $i \quad (2 \leq i \leq D)$. Pick vertices $x, y \in X$ such that $\delta(x, y) = i$, and pick $z \in C(x, y)$. Set

$$e_i := \Big| \{u | u \in X, \ xzuy \text{ is a kite of length } i\} \Big|,$$

and

$$f_i := \Big| \{u | u \in X, \ xzuy \text{ is a parallelogram of length } i\} \Big|.$$

(i) Suppose Γ is Q-polynomial with respect to the primitive idempotent

$$E_1 =\mid X \mid^{-1} \sum_{h=0}^{D} \theta_h^* A_h.$$

Then

$$f_i = \alpha_i e_i + \beta_i, \tag{7.15}$$

where

$$\alpha_i = \frac{\theta_2^* - \theta_1^*}{\theta_i^* - \theta_{i-1}^*}, \tag{7.16}$$

and

$$\beta_i = \frac{1}{\theta_i^* - \theta_{i-1}^*}(c_i(\frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*} + \theta_i^* - \theta_2^*) + c_{i-1}(\theta_{i-2}^* - \theta_i^*) + \theta_2^* - \theta_0^*). \tag{7.17}$$

(ii) Suppose Γ has classical parameters $(D,\ b,\ \alpha,\ \beta)$. Then (7.15) holds, where

$$\alpha_i = b^{i-2}, \tag{7.18}$$

$$\beta_i = 0. \tag{7.19}$$

Proof. (i) Define

$$x_y^- := \sum_{\substack{u \in X \\ \delta(x,u)=1 \\ \delta(u,y)=i-1}} \hat{u},$$

and

$$y_x^- := \sum_{\substack{u \in X \\ \delta(y,u)=1 \\ \delta(u,x)=i-1}} \hat{u}.$$

By Terwilliger[7, Theorem 3.3(vii)], we have

$$E_1(x_y^- - y_x^-) = c_i \frac{\theta_1^* - \theta_{i-1}^*}{\theta_0^* - \theta_i^*}(E_1\hat{x} - E_1\hat{y}), \tag{7.20}$$

so

$$\langle E_1(x_y^- - y_x^-), \hat{z}\rangle = c_i \frac{\theta_1^* - \theta_{i-1}^*}{\theta_0^* - \theta_i^*}\langle E_1\hat{x} - E_1\hat{y}, \hat{z}\rangle. \tag{7.21}$$

Evaluating the inner products in (7.21) using (7.14), we obtain

$$\begin{aligned} &\mid X \mid^{-1} (\theta_0^* + e_i\theta_1^* + (c_i - 1 - e_i)\theta_2^* - c_{i-1}\theta_{i-2}^* - f_i\theta_{i-1}^* - (c_i - c_{i-1} - f_i)\theta_i^*) \\ = &\mid X \mid^{-1} c_i \frac{(\theta_1^* - \theta_{i-1}^*)^2}{\theta_0^* - \theta_i^*}. \end{aligned} \tag{7.22}$$

Solving (7.22) for f_i we obtain (7.15).

(ii) By [2, p250], Γ is Q-polynomial with respect to a primitive idempotent

$$E = \mid X \mid^{-1} \sum_{h=0}^{D} \theta_h^* A_h,$$

where

$$\theta_j^* - \theta_0^* = (\theta_1^* - \theta_0^*) \begin{bmatrix} j \\ 1 \end{bmatrix} b^{1-j} \quad (0 \leq j \leq D). \tag{7.23}$$

In particular (7.15)-(7.17) hold by (i). Lines (7.18), (7.19) are obtained by eliminating θ_2^*, θ_{i-1}^*, θ_i^* in (7.16), (7.17) using (7.23), and simplifying using (7.8). This proves Lemma 7.3.

Proof of Theorem 7.2. The equivalence of (iii), (iv), (v) is from [9, Theorem 2.6].

(iv), (v)→(vi). This is clear.

(vi)→(iii). This immediate from Terwilliger[8, Theorem 2.11(ii)].

(iii), (vi)→(i). This is immediate from Lemma 7.3(ii).

(i)→(ii). This is clear.

(ii)→(iv). This is from Lemma 6.12.

Now we have the equivalence of (i), (ii), (iii), (iv), (v), (vi).

(i)→(vii). Γ is $(D-1)$-bounded by Theorem 6.4, so Γ is D-bounded by Lemma 6.2(iii).

(vii)→(viii). This is clear by Lemma 6.2(ii).

(viii)→(ii). This is clear by Lemma 6.5.

REFERENCES

1. E. Bannai, T. Ito, Algebraic Combinatorics: Association Schemes, Benjamin-Cummings Lecture Note 58. Menlo Park, 1984.
2. A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer Verlag, New York, 1989.
3. A. E. Brouwer, H. A. Wilbrink, The structure of near polygons with quads, Geom. Dedicata 14 (1983), 145-176
4. E. E. Shult, A. Yanushka, Near n-gons and Line Systems, Geom. Dedicata 9 (1980), 1-72.
5. A. A. Ivanov and S. V. Shpectorov, Characterization of the association schemes of Hermitian forms over $GF(2^2)$, Geom. Dedicata, 30 (1989), 23-33.

6. P. Terwilliger, The subconstituent algebra of an association scheme, I, J. Alg. Combin. 1, no.4 (1992), 363-388.
7. P. Terwilliger, A New Inequality for Distance-Regular Graphs, Discrete Mathematics, 137:319-332, 1995.
8. P. Terwilliger, Kite-Free Distance-Regular Graphs, European Journal of Combinatorics, 16(4):405-414, 1995.
9. C. Weng, Kite-Free P- and Q-Polynomial Schemes, Graphs and Combinatorics, 11:201-207, 1995.