Nontrivial Pooling Designs

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3. We arrange such a group testing design by the following binary matrix M.
4. Let M be the $t \times n$ binary matrix defined by

$$M_{ij} = \begin{cases} 1, & j \in T_i; \\ 0, & j \notin T_i \end{cases}$$

for $1 \leq i \leq t$ and $j \in [n]$.

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4. In the above definition, it suffices to assume $|P| = 1$.
5. M is d-disjunct if for any $d+1$ distinct columns M_{i_0}, M_{i_1}, $\ldots$, M_{i_d}, we have $M_{i_0} \not\subseteq \bigcup_{j=1}^{d} M_{i_j}$

Exercise

Show that a d-disjunct matrix is $\binom{[n]}{\leq d}$-disjunct.

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Exercise

Show that a W-disjunct matrix is W-decidable.

Problem

Find a W-decidable matrix which is not W-disjunct?

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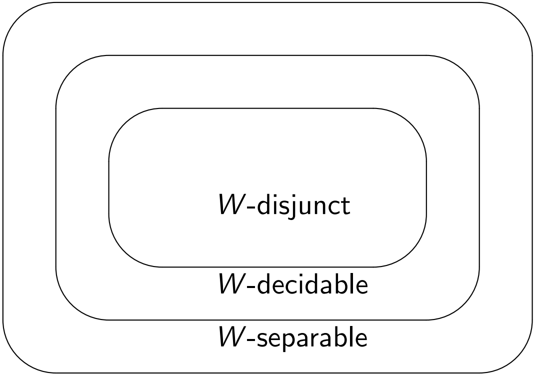
Exercise

A W-decidable matrix is W-separable for any $W \subseteq F_2^n$.

Relations



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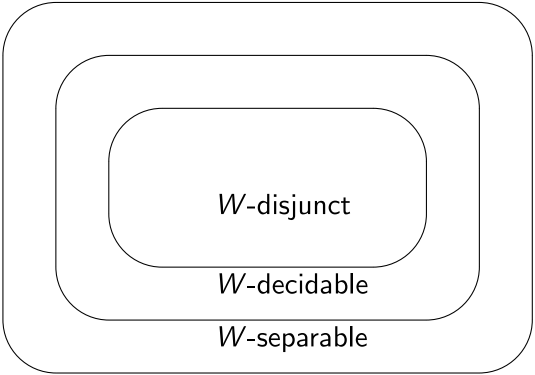

Find the relation between the above three classes of binary matrices with slightly changing W and possibly adding or deleting a few rows.

Note that for each $t \times n$ binary matrix M there exists a unique maximal $W_M \subseteq F_2^n$ such that M is W_M-decidable, in fact, $W_M = \{\mathbf{P} \in F_2^n \mid P = \bigcap_{o_M(\mathbf{P})_i = 0} \overline{T_i}\}$.

A property distinguishes decidable matrix from others

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Problem

Study the map $M \to W_M$.

1-disjunct matrix

Example

A 1-disjunct matrix to detect the infected item **3** from $\{1, 2, \mathbf{3}, 4, 5, 6\}$:

Tests/Items	1	2	**3**	4	5	6		$o_M(\{\mathbf{3}\})$
one	1	1	1	0	0	0	$\rightarrow$	1
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In fact the above 4×6 matrix M has $W_M =$
$\binom{[6]}{\leq 1} \cup \{\{3, 5, 6\}, \{2, 4, 6\}, \{1, 4, 5\}, \{4, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$.

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In applying to a group testing, we need the number t of tests is smaller than the number n of items, otherwise we would rather test the items one by one.

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The first q which is not a prime power is when $q = 6$.

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7. The case $r = 6$ has been ruled out by Bruck-Ryser-Chowla theorem.
8. The next case $r = 10$ has been ruled out by massive computer calculations.
9. There is nothing more known, in particular $r = 12$ is still open.

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Note that if there exists a nontrivial d-disjunct matrix with $(d+1)^2 - 1$ rows then EFF's conjecture is false. See page 29 of the book "Pooling Designs and nonadaptive group testing" by Ding-Zhu Du and Frank K. Hwang for a description of EFF's conjecture.

A 36 × 37 5-disjunct matrix

吴欣融的答案 10/29/01

以11,12,…,16,21,…,26,…,66代表36个元素。
以下之37个子集任二子集最多有一共同元素，
故每一子集会被任意其它五个子集的联集覆盖。

①	11	22	33	54	65	26	⑲	11	52	23	64	45	56
②	12	23	34	55	66	21	⑳	12	53	24	65	46	51
③	13	24	35	56	61	22	㉑	13	54	25	66	41	52
④	14	25	36	51	62	23	㉒	14	55	26	61	42	53
⑤	15	26	31	52	63	24	㉓	15	56	21	62	43	54
⑥	16	21	32	53	64	25	㉔	16	51	22	63	44	55
⑦	11	32	43	24	55	36	㉕	11	62	53	44	35	66
⑧	12	33	44	25	56	31	㉖	12	63	54	45	36	61
⑨	13	34	45	26	51	32	㉗	13	64	55	46	31	62
⑩	14	35	46	21	52	33	㉘	14	65	56	41	32	63
⑪	15	36	41	22	53	34	㉙	15	66	51	42	33	64
⑫	16	31	42	23	54	35	㉚	16	61	52	43	34	65
⑬	11	42	63	34	25	46	㉛	11	21	31	41	51	61
⑭	12	43	64	35	26	41	㉜	12	22	32	42	52	62
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4. A subset $T \subseteq \mathbb{Z}_m \times F_q$ is said to have the forward difference distinct property in $\mathbb{Z}_m \times F_q$ if the set

$$D_T := \{(j, y) - (i, x) \mid (i, x), (j, y) \in T \text{ with } i < j\}$$

consists of $\frac{|T|(|T|-1)}{2}$ elements.

The Set ${}_mT_q$

Let ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ be defined by

$$ {}_mT_q = \{(i, a^i) \mid i \in \mathbb{Z}_m, 0 \leq i \leq q-1\}. $$

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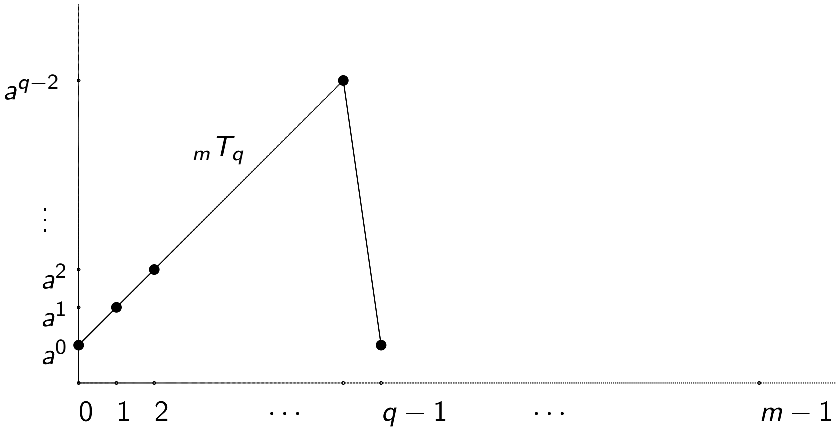
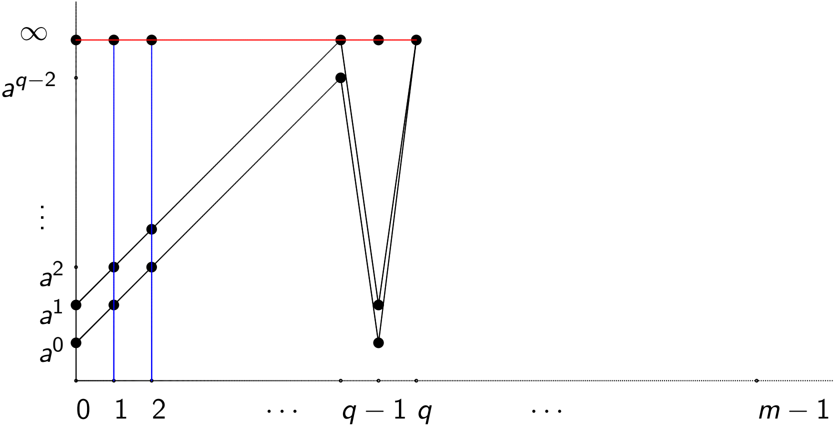


A preview of the finial result



Lines in $Z_m \times (F_q \cup \{\infty\})$

The Set ${}_5T_5$

For $q = 5$, $a = 2$,

$$ {}_5T_5 = \{(0,1),(1,2),(2,4),(3,3),(4,1)\} $$

and

$$ \begin{aligned} D_{{}_5T_5} = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\ & (2,3),(2,1),(2,2) \\ & (3,2),(3,4) \\ & (4,0) \quad \}. \end{aligned} $$

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Since $|D_{{}_5T_5}| = 10$, the set ${}_5T_5$ has the forward difference distinct property in $\mathbb{Z}_5 \times F_5$.

Difference Property

A subset $T \subseteq \mathbb{Z}_m \times F_q$ is said to have the difference distinct property in $\mathbb{Z}_m \times F_q$ if the set $-D_T \cup D_T$ consists of $|T|(|T|-1)$ elements.

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From the structure of $D_{{}_mT_q}$ we find $(0,x) \notin -D_{{}_mT_q} \cup D_{{}_mT_q}$ for any $x \in F_q$. This property will be used later.

Non-example

We have seen

$$\begin{aligned} D_{5\,T_5} = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\ & (2,3),(2,1),(2,2) \\ & (3,2),(3,4) \\ & (4,0) \quad \}. \end{aligned}$$

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Hence

$$\begin{aligned} -D_{5}\,T_5 \quad = \{ \quad & (4,4),(4,3),(4,1),(4,2) \\ & {\color{red}(3,2),(3,4)},(3,3) \\ & {\color{red}(2,3),(2,1)} \\ & (1,0) \quad \}. \end{aligned}$$

Non-example

We have seen

$$D_{_5T_5} = \{ \begin{array}{l} (1,1),(1,2),(1,4),(1,3) \\ (2,3),(2,1),(2,2) \\ (3,2),(3,4) \\ (4,0) \end{array} \}.$$

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Since $|-D_{_5T_5} \cup D_{_5T_5}| = 16 \neq 20$, the set $_5T_5$ does not have the difference distinct property in $\mathbb{Z}_5 \times F_5$.

Embedding

For positive integers $n < m$, the set $\mathbb{Z}_n$ can be viewed as a subset of $\mathbb{Z}_m$ in the usual way. Hence we have $\mathbb{Z}_n \times F_q \subseteq \mathbb{Z}_m \times F_q$.

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$$\begin{aligned} D_{6}{}_{T_5} = D_{5}{}_{T_5} \quad = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\ & (2,3),(2,1),(2,2) \\ & (3,2),(3,4) \\ & (4,0) \quad \}. \end{aligned}$$

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$$
\begin{aligned}
D_{{}_6T_5} = D_{{}_5T_5} = \{ \quad & (1,1),(1,2),(1,4),(1,3) \\
& (2,3),(2,1),(2,2) \\
& (3,2),(3,4) \\
& (4,0) \quad \}.
\end{aligned}
$$

Hence considering as the negative in $\mathbb{Z}_6 \times F_5$, we have

$$
\begin{aligned}
-D_{{}_6T_5} = \{ \quad & (5,4),(5,3),(5,1),(5,2) \\
& (4,2),(4,4),(4,3) \\
& (3,3),(3,1) \\
& (2,0) \quad \}.
\end{aligned}
$$

Since $|-D_{{}_6T_5} \cup D_{{}_6T_5}| = 20$ now, the set ${}_6T_5$ has the difference distinct property in $\mathbb{Z}_6 \times F_5$.

Problem

Determine the prime power integer q such that with a suitable choice of a generator $a \in F_q$, the set ${}_{q+1}T_q$ has the difference distinct property in $\mathbb{Z}_{q+1} \times F_q$.

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By direct computing by hands, we find the above statement is true for $q = 2, 4, 5$ and is false for $q = 3, 7$ (First two primes in $4k + 3$ form).

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Example

Note that

$$ {}_4T_3 \quad = \quad \{(0,1),(1,2),(2,1)\}, $$

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Example

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Hence the set ${}_4T_3$ does not have the difference distinct property in $\mathbb{Z}_4 \times F_3$.

${}_mT_q$ has the forward difference distinct property

Theorem

The set ${}_mT_q$ has the forward difference distinct property in $\mathbb{Z}_m \times T_q$.

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Suppose for $0 \leq i < j \leq q - 1$ we have $j - i = c$ and $a^j - a^i = d$.

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Suppose for $0 \le i < j \le q-1$ we have $j - i = c$ and $a^j - a^i = d$. Note that $1 \le c \le q-1$. If $c = q-1$ then $j = q-1$ and $i = 0$. If $c \ne q-1$ then $a^i = d/(a^{j-i} - 1) = d/(a^c - 1)$ and $j = c + i$.

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${}_{2q-1}T_q$ has the difference distinct property

Theorem

For $m \geq 2q-1$*, the set* ${}_mT_q$ *has the difference distinct property in* $\mathbb{Z}_m \times T_q$.

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Proof.

By the theorem in the last page we have
$|D_{{}_mT_q}| = |-D_{{}_mT_q}| = q(q-1)/2$.

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Theorem

For $m \geq 2q-1$, the set ${}_mT_q$ has the difference distinct property in $\mathbb{Z}_m \times T_q$.

Proof.

By the theorem in the last page we have $|D_{{}_mT_q}| = |-D_{{}_mT_q}| = q(q-1)/2$. The first coordinate of an element in $D_{{}_{2q-1}T_q}$ runs from 1 to $q-1$, and the first coordinate of an element in $-D_{{}_{2q-1}T_q}$ from $m+1-q$ to $m-1$.

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Lines with any two intersecting in at most a point

Theorem

Suppose that ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ *has the difference distinct property in* $\mathbb{Z}_m \times F_q$*. Set* $\mathcal{B} = \{u +_m T_q \mid u \in \mathbb{Z}_m \times F_q\}$*. Then* $|L \cap L'| \leq 1$ *for any distinct* $L, L' \in \mathcal{B}$*.*

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Proof.

Routine. □

Lines with any two intersecting in at most a point

Theorem

Suppose that ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ has the difference distinct property in $\mathbb{Z}_m \times F_q$. Set $\mathcal{B} = \{u +_m T_q \mid u \in \mathbb{Z}_m \times F_q\}$. Then $|L \cap L'| \leq 1$ for any distinct $L, L' \in \mathcal{B}$.

Proof.

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An element in $\mathcal{B}$ is called a line and an element in $Z_m \times F_q$ is called a point.

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Proof.

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An element in $\mathcal{B}$ is called a line and an element in $Z_m \times F_q$ is called a point. Note that there are mq lines and mq points, and a line has $q = |T|$ points with q different first coordinates.

Lines with any two intersecting in at most a point

Theorem

Suppose that $_mT_q \subseteq \mathbb{Z}_m \times F_q$ has the difference distinct property in $\mathbb{Z}_m \times F_q$. Set $\mathcal{B} = \{u +_m T_q \mid u \in \mathbb{Z}_m \times F_q\}$. Then $|L \cap L'| \leq 1$ for any distinct $L, L' \in \mathcal{B}$.

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Routine. □

An element in $\mathcal{B}$ is called a line and an element in $Z_m \times F_q$ is called a point. Note that there are mq lines and mq points, and a line has $q = |T|$ points with q different first coordinates. Apparently more lines can be added to $\mathcal{B}$ still having the conclusion of the above theorem, for example, adding vertical lines to $\mathcal{B}$.

Adding an infinity point to each line

As previous page, assume that any two lines in $\mathcal{B} = \{u +_m T_q \mid u \in \mathbb{Z}_m \times F_q\}$ intersect at at most one point.

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Since $(0, x) \notin -D_{{}_mT_q} \cup D_{{}_mT_q}$, we have $L \cap ((0, x) + L) = \emptyset$ for any nonzero $x \in F_q$ and $L \in \mathcal{B}$.

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Since $(0, x) \notin -D_{{}_m T_q} \cup D_{{}_m T_q}$, we have $L \cap ((0, x) + L) = \emptyset$ for any nonzero $x \in F_q$ and $L \in \mathcal{B}$. Then $\mathcal{B}$ is partitioned into m classes with each class consisting of parallel lines (non-intersecting lines). We add a common point (i, ∞) to each line in a parallel class where $i \in \mathbb{Z}_m$ is not appearing in the first coordinate of any points of the line and $i - 1$ appearing in some point of the line. This forms a new set $\mathcal{B}'$ of Lines with underground point set $Z_m \times (F_q \cup \{\infty\})$. Note that any two distinct lines in $\mathcal{B}'$ intersect in at most one point too.

Vertical Lines and infinite line

Set $V_i = \{(i, j) \mid j \in F_q \cup \{\infty\}\ \}$ for $0 \leq i \leq m - 1$, and V_i is called the ith vertical line. Set $H = \{(i, \infty) \mid 0 \leq i \leq q\}$ (here assuming $m > q$), and H is called an infinite line.

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Set $\mathcal{B}'' := \mathcal{B}' \cup \{H, V_0, V_1, \ldots, V_{m-1}\}$. Then $|Z_m \times (F_q \cup \{\infty\})| = m(q+1)$ and $|\mathcal{B}''| = m(q+1)+1$.

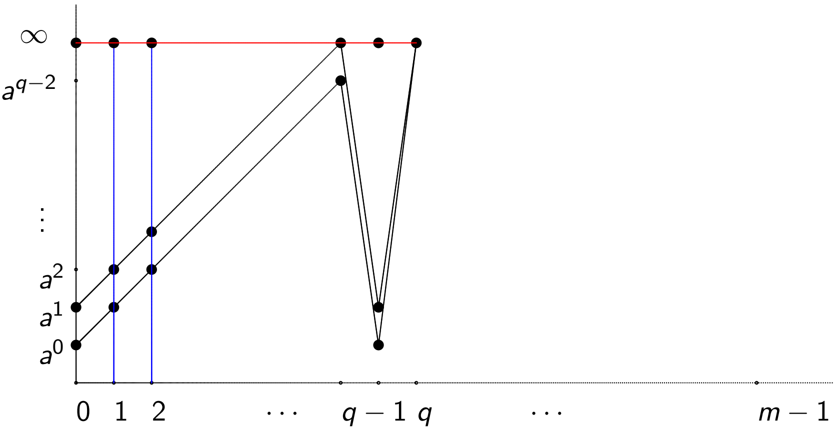
Conclusion

Suppose that ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ has the difference distinct property in $\mathbb{Z}_m \times F_q$, for example in the case $m \geq 2q - 1$ or $m = q + 1 = 6$.

Conclusion

Suppose that ${}_mT_q \subseteq \mathbb{Z}_m \times F_q$ has the difference distinct property in $\mathbb{Z}_m \times F_q$, for example in the case $m \geq 2q - 1$ or $m = q + 1 = 6$. Let M be the incidence matrix of $\mathbb{Z}_m \times (F_q \cup \{\infty\})$ and $\mathcal{B}''$. Then M is a nontrivial q-disjunct matrix with $m(q + 1)$ rows.

A Review of our result



Lines in $Z_m \times (F_q \cup \{\infty\})$

The end

Thank you for your attention.