Korea-Japan Workshop on Algebra and Combinatorics

The flipping puzzle on a simple graph

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Flipping Puzzle

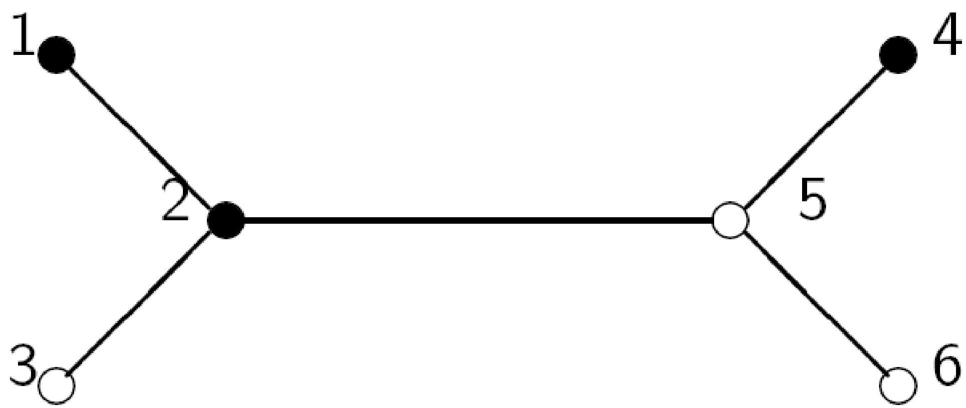
Let $X = (V, E)$ be a finite simple connected graph with $|V| = n$. A **configuration** of X is an assignment $f : V \longrightarrow \{0, 1\}$, and is viewed as a column vector indexed by V.

0=white, 1=black

A Move in the Puzzle

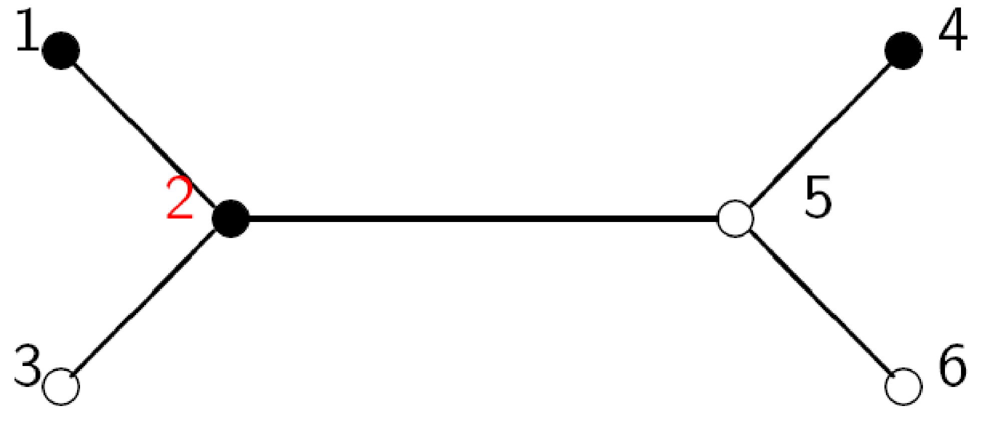
A **move** is to select one vertex $v \in V$ having black state in the configuration f and then change those states of all neighbors of v to become a new configuration. This is the flipping puzzle on X.

Example of a Configuration



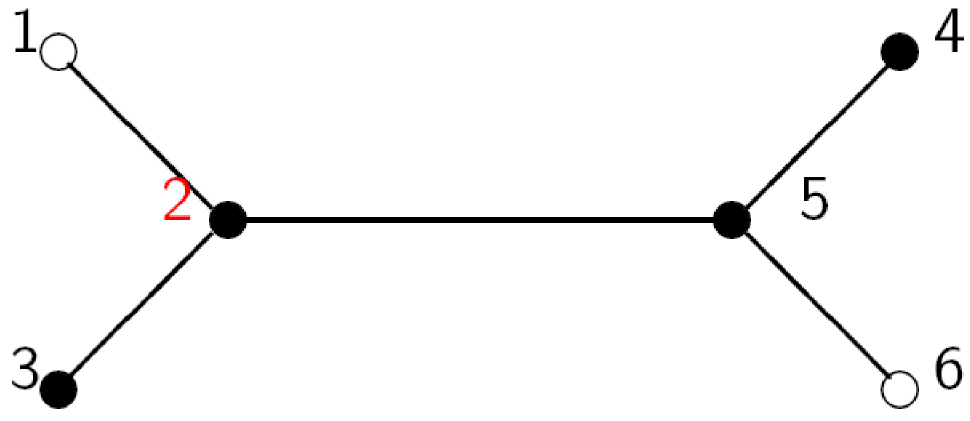
$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

A Move by Selecting Vertex 2



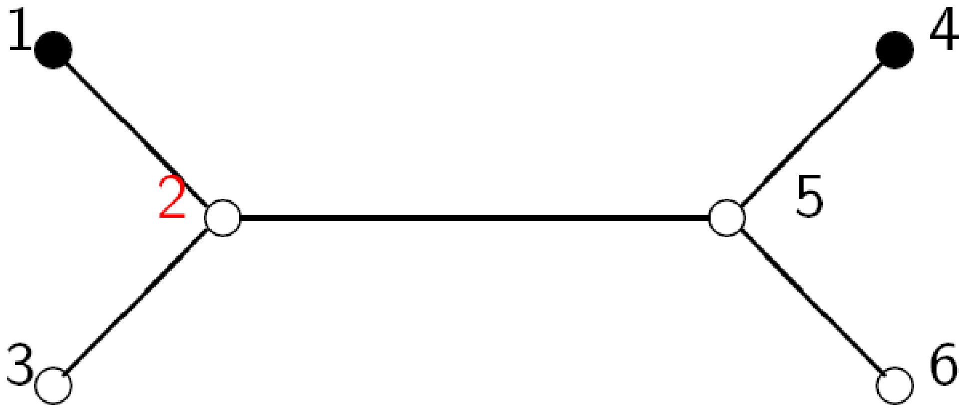
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = ?$$

New Configuration after the Move



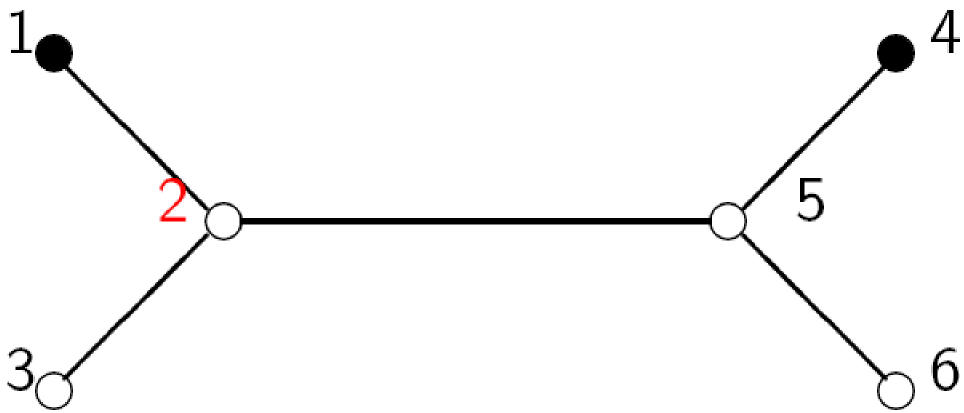
$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Feigning Move



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = ?$$

No Effect after a Feigning Move



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Flipping Classes

- Two configurations are **equivalent** if one is obtained from the other by a move
- An orbit of configurations is called a **flipping class**

History

- When X is Dynkin Diagram, a configuration is called a **Vogan diagram** with identity graph involution.
- Each Vogan diagram corresponds to a real simple Lie algebra.
- Equivalent Vogan diagrams represent the same real simple Lie algebra.

Problem

For each graph X, determine its flipping classes

Previous Results

Flipping classes of Dynkin Diagrams and extended Dynkin diagrams are determined by Meng-Kiat Chuah and Chu-Chin Hu in 2004[4], 2006[5]

Other Graphs that Flipping Classes Are Completely Determined

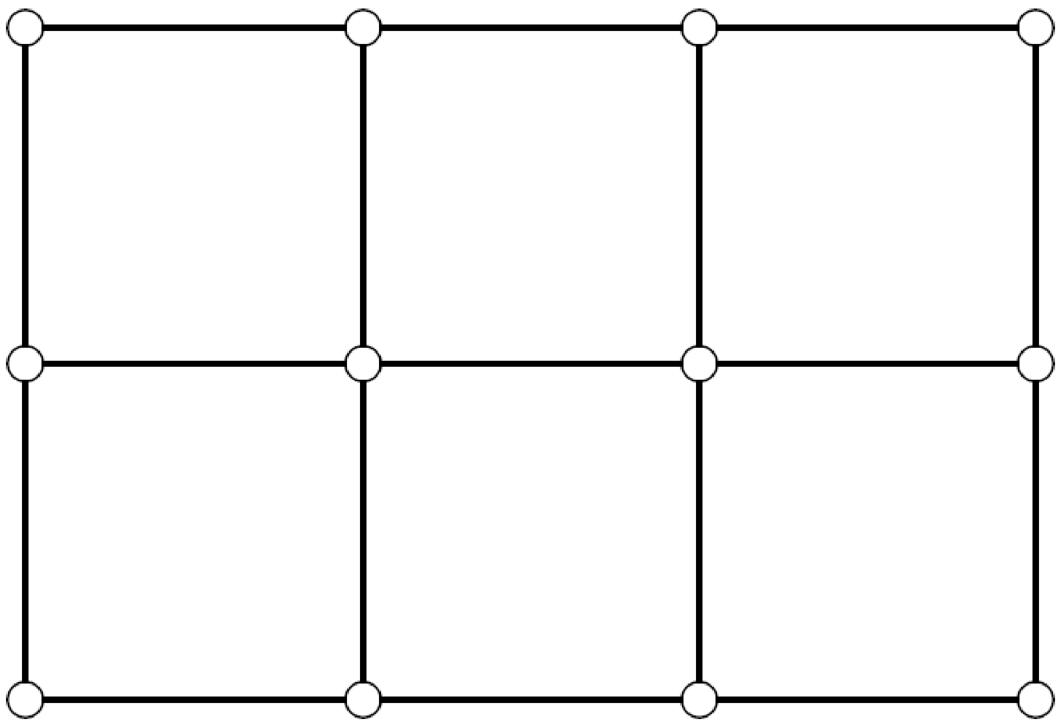
- A graph with n vertices which contains an induced path of n-1 vertices (H. Huang and ---2008[2])
- Line graphs (Wu[8], H. Huang and ---2008[3])

The Number of Flipping classes in a Tree with perfect matching

If X is a tree with a perfect matching, not a path, then there are exactly 3 flipping classes (Hau-wen Huang, preprint).

Open Problem

Determine the flipping classes of X when X is a chessboard.



Maximum-orbit-weight

For $u \in F_2^n$, let $w(u)$ denotes the Hamming weight of u, and for an flipping class O of X, $w(O) := \min\{w(u) \mid u \in O\}$ is called the weight of the flipping class O. The number

$$M(X) := \max\{w(O) \mid O \in \mathcal{P}\}$$

is called the maximum-orbit-weight of the graph S.

Borel–de Siebenthal Theorem

If X is a Dynkin diagram then $M(X) = 1$.

Recent Results of M(X)

X. Wang, Y. Wu [6] and H. Wu, G. J. Chang [7] independently show $M(X) \leq \lceil \ell/2 \rceil$ if X is a tree with ℓ leaves.

Y. Wu discovers that if X is the line graph of a simple graph Γ, then there is a close connection between $M(X)$ and the edge isoperimetric number of Γ [8].

$M(X) = 1$ if X is a tree with a perfect matching (Hua-wen Huang, preprint).

Open Problem

Determine all graphs X with $M(X) = 1$.

Variations of Flipping Puzzle

- σ -game (also can move on a white vertex)
- σ -plus-game (the state of selected vertex is also switched)
- Reeder's puzzle (dual version)

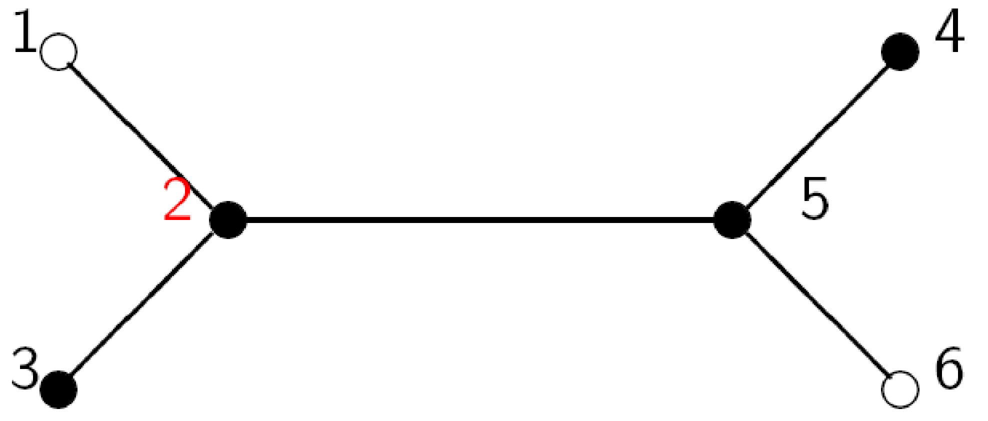
Associate a Move with a Matrix

For $s \in V$, we associate a matrix $\mathbf{s} \in \mathrm{Mat}_n(F_2)$, denoted by the bold type of s, as

$$\mathbf{s}_{uv} = \begin{cases} 1, & \text{if } u = v \text{, or } v = s \text{ and } uv \in E; \\ 0, & \text{else,} \end{cases}$$

where $u, v \in V$.

Matrix Associated with Move 2 (Revisited)



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Flipping Group of X

Lemma

For $s \in V$, $\mathbf{s}$ is an involution; that is, $\mathbf{s}^2 = I$, the identity matrix. In particular, $\mathbf{s} \in \mathrm{GL}_n(F_2)$. $\square$

Definition

Let $\mathbf{W}$ denote the subgroup of $\mathrm{GL}_n(F_2)$ generated by the set $\{\mathbf{s} \mid s \in V\}$. $\mathbf{W}$ is referring to the flipping group of X.

- Flipping group is a matrix group.
- Are there any objects related to flipping groups?

Coxeter Group of a graph X

Let $X = (V, E)$ be a finite simple connected graph. For $s, s' \in V$, set

$$m(s, s') = \begin{cases} 3, & \text{if } ss' \in E; \\ 2, & \text{if } s \neq s', ss' \notin E; \\ 1, & \text{if } s = s'. \end{cases}$$

A (simple laced) **Coxeter group** associated with X is a group $W = W(V, m)$ with the finite set V as generators subject only to the relations $(ss')^{m(s,s')} = 1$ for $s, s' \in V$.

Notation Reminding

W is the Coxeter group of X and the bold type $\mathbf{W}$ is the flipping group of X. For a vertex s in V, the bold type $\mathbf{s}$ in $\mathbf{W}$ associated with the move by selecting s.

Another Matrix associated with a vertex in V

Let $X = (V, E)$ be a simple connected graph with $|V| = n$. For $s \in V$ define an $n \times n$ matrix $\widetilde{s}$ over $\mathbb{R}$ by

$$\widetilde{s}_{uv} = \begin{cases} -1, & \text{if } u = v = s; \\ 1, & \text{if } u = v \neq s, \text{ or } u = s \text{ and } uv \in E; \\ 0, & \text{else.} \end{cases}$$

The coefficients of $\widetilde{s}$ are in fact in $\mathbb{Z}$.

Geometric Representation W

Then the map $s \rightarrow \widetilde{s}$ extends to a faithful representation of Coxeter group W into $n \times n$ matrices over $\mathbb{R}$.
(p. 113 in Humphreys's book "Reflection Groups and Coxeter Groups.")

It turns out that $\widetilde{s} \pmod 2$ is the transpose of $\mathbf{s}$.

Irreducible Graphs

We say a graph X is **irreducible** (resp. **reducible**) if $\det(A) = 1$ (resp. $\det(A) = 0$) in F_2, where A is the adjacency matrix of X.

Results on the Flipping Group **W**

1. The center $Z(\mathbf{W})$ of $\mathbf{W}$ is trivial (Huang, —[1]).
2. $W/Z(W)$ is isomorphic to $\mathbf{W}$ if X is an Dynkin diagram; moreover $|Z(W)| = 1$ or 2 (Huang, —[1]).
3. Among all n-vertex graphs, each of which contains an induced $(n-1)$-vertex path, there are at most $n-1$ flipping groups up to isomorphism (Huang, —[2]).
4. $\mathbf{W}$ is irreducible iff X is irreducible (preprint).
5. If X is reducible, then $\mathbf{W}$ is not completely reducible (preprint).
6. $\mathbf{W}$ is isomorphic to the symmetric group S_n when X is the line graph of a tree with n vertices (2009 Y. Wu[8]).

Results on the Flipping Group **W**

If X is the line graph of a graph with m edges and n vertices, then **W** is isomorphic to

$$(\mathbb{Z}/2\mathbb{Z})^{(n-1)(m-n+1)} \rtimes S_n \text{ if } n \text{ is odd;}$$
$$(\mathbb{Z}/2\mathbb{Z})^{(n-2)(m-n+1)} \rtimes S_n \text{ if } n \text{ is even}$$

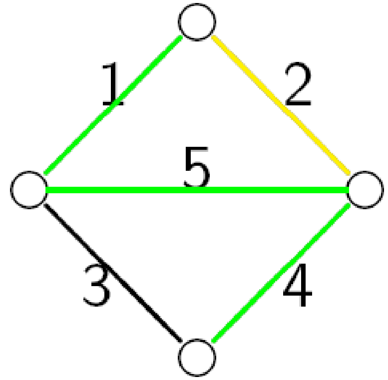
Idea in the Study

- We choose a nice set of new basis (simple basis) such the moves on this set like permutations of the set.

Simple Basis of a Path

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

The simple bases of line graphs:



$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Problems for Further Study

- For any graph X, what is the possible choice of a simple basis of X?
- Classify all flipping groups.
- Determine the minimum number of moves from a given configuration to another.
- Determine the minimum number of length when an element in the flipping group is written as a product of moves.
- Study the links between flipping puzzle and its variations.

References

- [1] Hau-wen Huang and Chih-wen Weng, Combinatorial representations of Coxeter groups over a field of two elements, arXiv:0804.2150, 14 Apr., 2008.
- [2] Hau-wen Huang and Chih-wen Weng, The flipping puzzle of a graph, arXiv:0808.2104, 15 Aug., 2008.
- [3] Hau-wen Huang and Chih-wen Weng, The flipping group of a line graph, arXiv:0809.4399, 25 Sep., 2008.
- [4] Meng-Kiat Chuah and Chu-Chin Hu, Equivalence classes of Vogan diagrams, Journal of Algebra, 279(2004), 22--37.
- [5] Meng-Kiat Chuah and Chu-Chin Hu, Extended Vogan diagrams, Journal of Algebra, 301(2006), 112--147.
- [6] Xinmao Wang and Yaokun Wu, Minimum light number of lit-only σ-game on a tree, Theoretical Computer Science} 381 (2007) {292--300}.
- [7] Hsin-Jung Wu and Gerard J. Chang, A study on equivalence classes of painted graphs, Master Thesis, NTU, Taiwan, 2006.
- [8] Yaokun Wu, Lit-only sigma game on a line graph, European Journal of Combinatorics}, 30 (2009) 84--95.

Thank You for Your Attention