D-bounded Property in a Distance-regular Graph with $a_1=0$ and $a_2 \neq 0$

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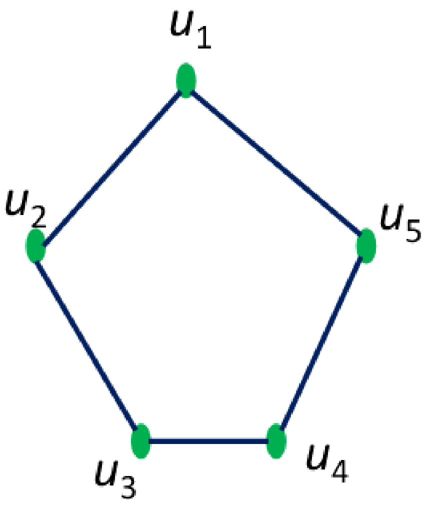
Let $\Gamma=(X, R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter $D:=\max\{\partial(x,y) \mid x,y \in X\}$.

By a **pentagon**, we mean a 5-tuple $u_1 u_2 u_3 u_4 u_5$ consisting of distinct vertices in Γ such that $\partial(u_i, u_{i+1}) = 1$ for $1 \leq i \leq 4$ and $\partial(u_5, u_1) = 1$.



A graph Γ is said to be **distance-regular** whenever for all integers $0 \leq h, i, j \leq D$, and all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\Gamma_i(x) \cap \Gamma_j(y)|$$

is independent of x, y.

For two vertices $x, y \in X$, with $\partial(x, y) = i$, set

$$\begin{aligned} B(x, y) &:= \Gamma_1(x) \cap \Gamma_{i+1}(y), \\ C(x, y) &:= \Gamma_1(x) \cap \Gamma_{i-1}(y), \\ A(x, y) &:= \Gamma_1(x) \cap \Gamma_i(y). \end{aligned}$$

Note that

$$
\begin{aligned}
b_i := |B(x, y)| &= p^i_{1\ i+1}, \\
c_i := |C(x, y)| &= p^i_{1\ i-1}, \\
a_i := |A(x, y)| &= p^i_{1\ i}
\end{aligned}
$$

are independent of x, y.

Recall that a sequence x, z, y of vertices of Γ is *geodetic* whenever

$$\partial(x, z) + \partial(z, y) = \partial(x, y).$$

A sequence x, z, y of vertices of Γ is **weak-geodetic** whenever

$$\partial(x,z) + \partial(z,y) \leq \partial(x,y) + 1.$$

Definition

A subset $\Delta \subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y \in \Delta \Longrightarrow z \in \Delta.$$

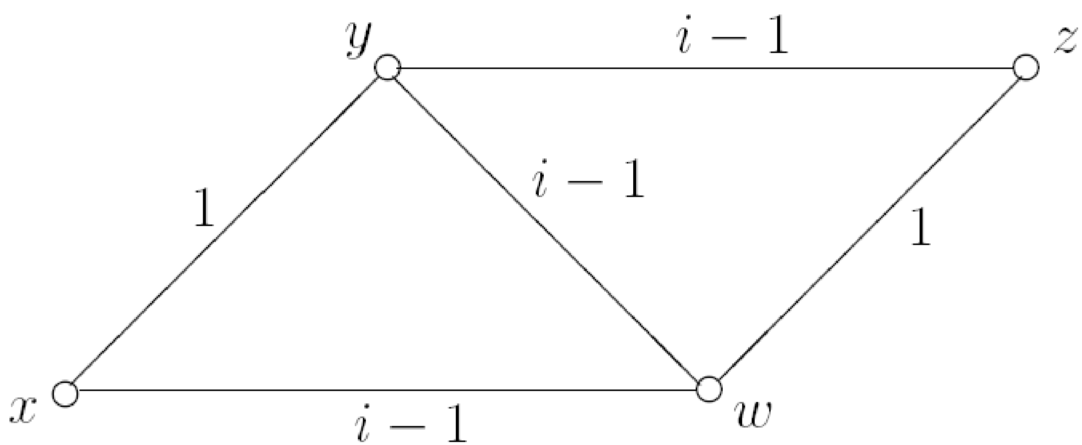
Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in (Suzuki, On strongly closed subgraphs of highly regular graphs, European J. Combin., 16(1995), 197–220).

Definition

Γ is said to be *i-bounded* whenever for all $x, y \in X$ with $\partial(x, y) \leq i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x, y)$ which contains x and y.

Note that a $(D-1)$-bounded distance-regular graph is clear to be D-bounded.

By a **parallelogram of length** i, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x,y) = \partial(z,w) = 1$, $\partial(x,z) = i$, and $\partial(x,w) = \partial(y,z) = \partial(y,w) = i-1$.



Theorem

Let Γ denote a distance-regular graph with diameter $D \geq 3$, and intersection numbers $a_1 = 0$, $a_2 \neq 0$. Fix an integer $1 \leq d \leq D - 1$ and suppose Γ contains no parallelograms of any length up to $d + 1$. Then Γ is d-bounded.

Applying our main Theorem with previous results, we have

Theorem

Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose the intersection number $a_2 \neq 0$. Fix an integer $2 \leq d \leq D-1$. Then the following two conditions (i), (ii) are equivalent:

(i) Γ *is* d*-bounded.*

(ii) Γ *contains no parallelograms of any length up to* $d+1$ *and* $b_1 > b_2$.

□

A subset Ω of X is **weak-geodetically closed with respect to a vertex** $x \in \Omega$ if and only if

$$C(y,x) \subseteq \Omega \quad \text{and} \quad A(y,x) \subseteq \Omega \qquad \text{for all} \quad y \in \Omega.$$

Note that Ω is weak-geodetically closed if and only if for any vertex $x \in \Omega$, Ω is weak-geodetically closed with respect to x.

Proof of the Theorem

Known Results

Theorem

Let Γ *be a distance-regular graph with diameter* $D \geq 3$. *Let* Ω *be a regular subgraph of* Γ *with valency* γ *and set* $d := \min\{i \mid \gamma \leq c_i + a_i\}$. *Then the following (i),(ii) are equivalent.*

(i) Ω *is weak-geodetically closed with respect to at least one vertex* $x \in \Omega$.

(ii) Ω *is weak-geodetically closed with diameter* d.

In this case $\gamma = c_d + a_d$. □

(—, Weak-geodetically closed subgraphs in distance-regular graphs, Graphs and Combinatorics, 14(1998), 275–304.)

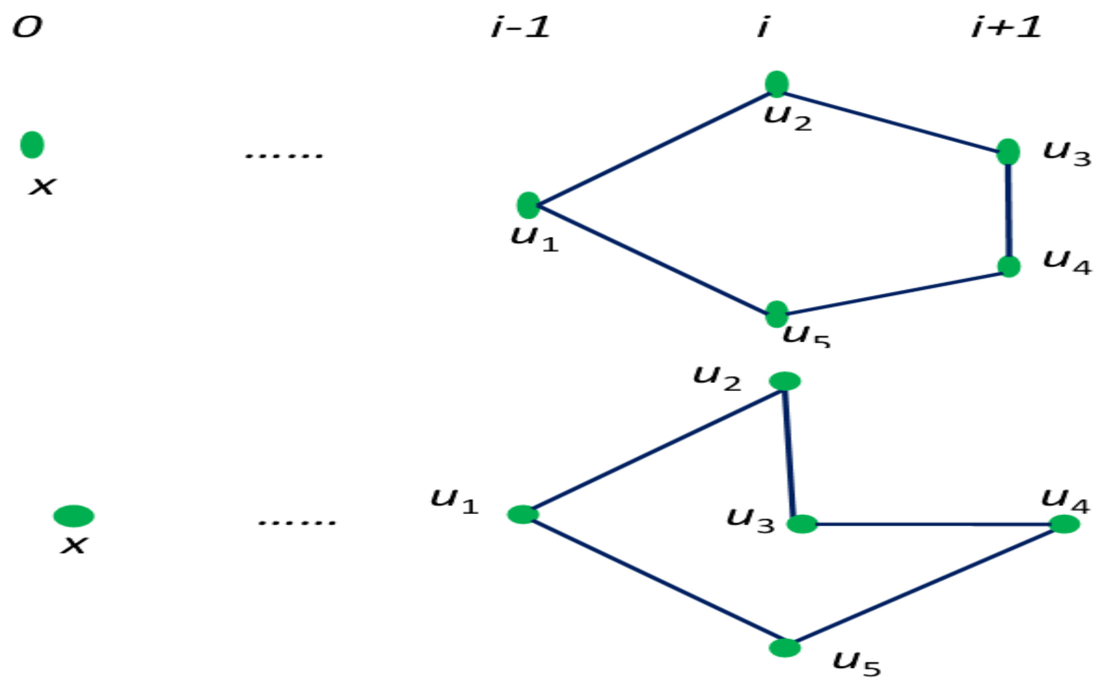
Theorem

Let Γ be a distance-regular graph with diameter $D \geq 3$. Suppose $b_1 > b_2$, $a_2 \neq 0$, and Γ contains no parallelograms of length up to 3. Then Γ is 2-bounded. □

(—, Weak-geodetically closed subgraphs in distance-regular graphs(Proposition 6.7), Graphs and Combinatoric, 14(1998), 275–304, and H. Suzuki, Strongly closed subgraphs of a distance-regular graph with geometric girth five(Theorem 1.1), Kyushu Journal of Mathematics, 50(2)(1996), 371–384.)

Definition

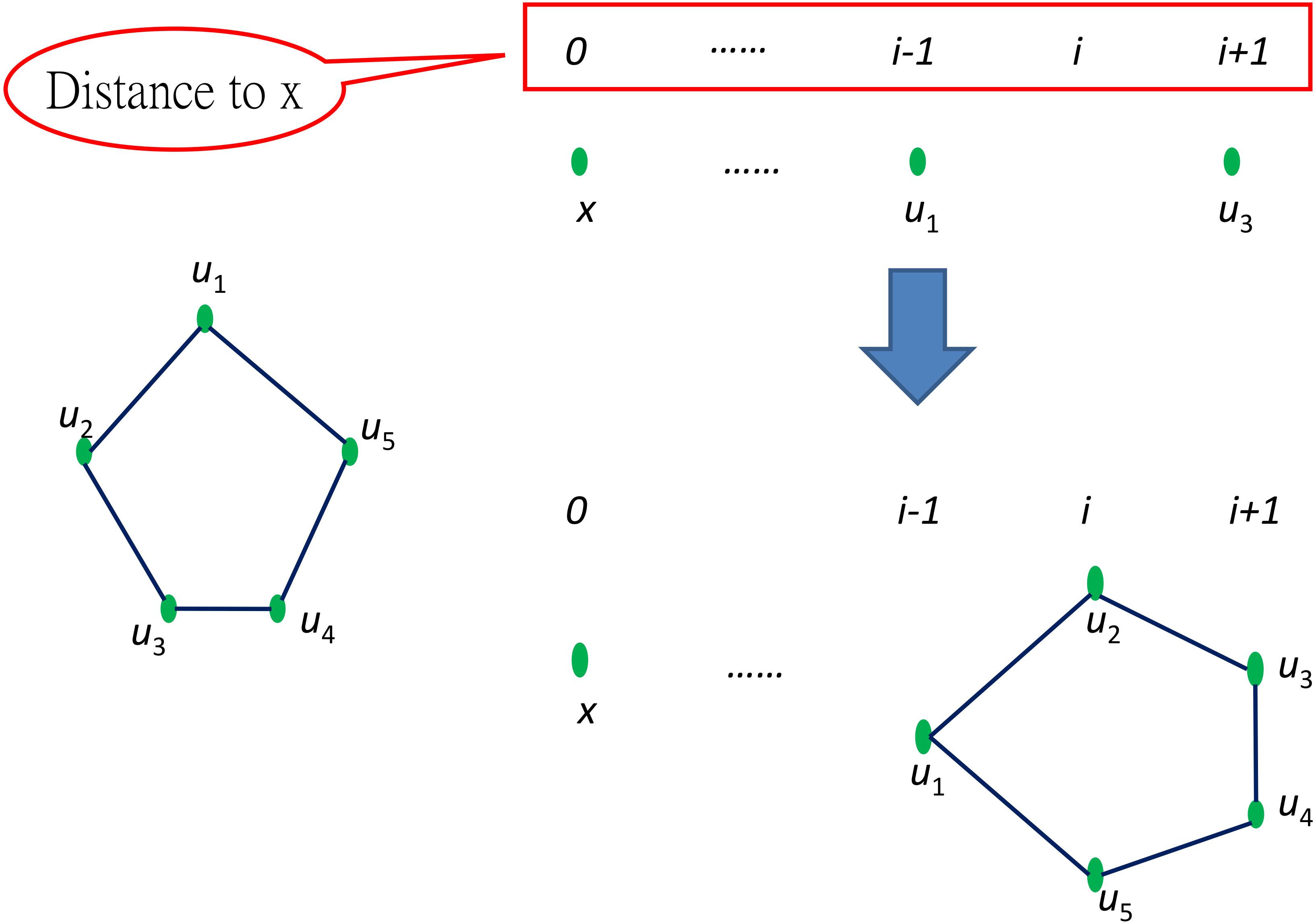
Fix a vertex $x \in X$. A pentagon $u_1u_2u_3u_4u_5$ has *shape* i_1, i_2, i_3, i_4, i_5 *with respect to* x if $i_j = \partial(x, u_j)$ for $1 \leq j \leq 5$.

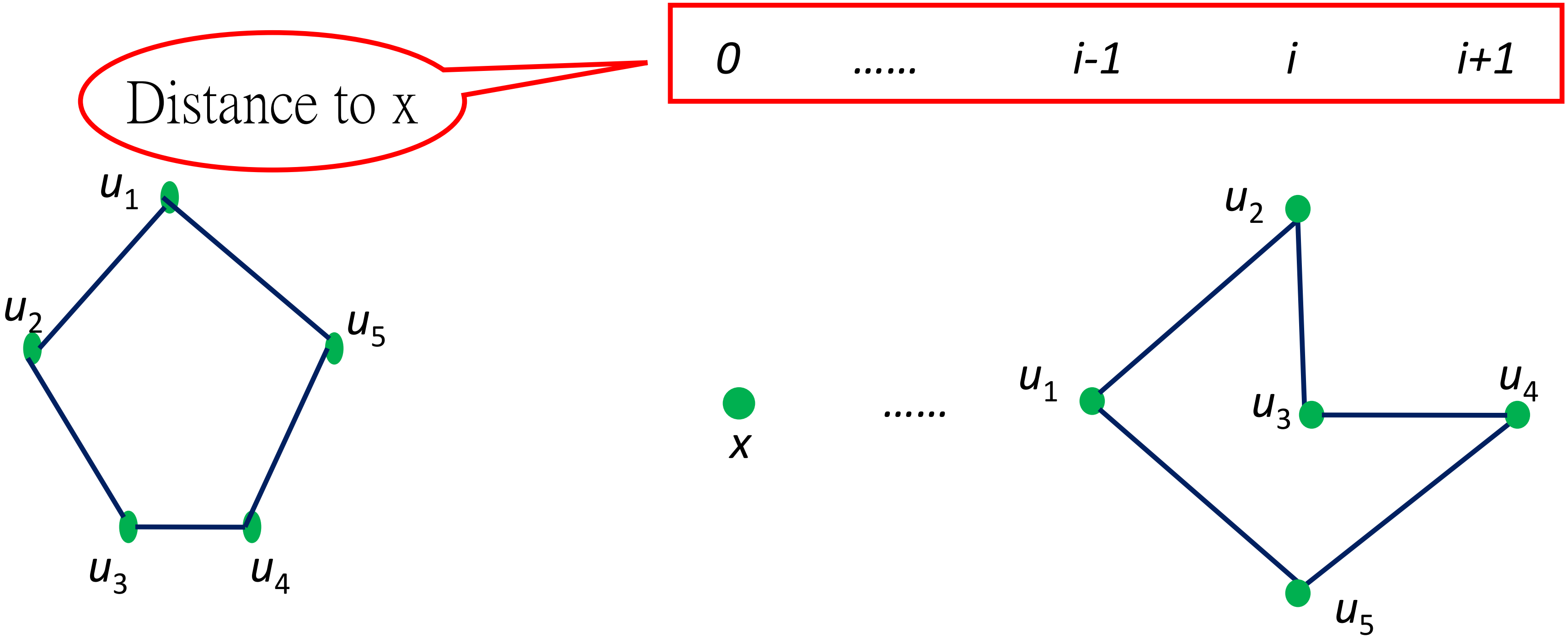


Theorem

Let Γ *be a distance-regular graph with diameter* $D \geq 3$*. Suppose* $a_1 = 0$*,* $a_2 \neq 0$ *and* Γ *contains no parallelograms of length up to* $d+1$ *for some integer* $d \geq 2$*. Let* x *be a vertex of* Γ*, and let* $u_1u_2u_3u_4u_5$ *be a pentagon of* Γ *such that* $\partial(x, u_1) = i-1$ *and* $\partial(x, u_3) = i+1$ *for* $1 \leq i \leq d$*. Then the pentagon* $u_1u_2u_3u_4u_5$ *has shape* $i-1, i, i+1, i+1, i$ *with respect to* x*.* □

(—, Weak-geodetically closed subgraphs in distance-regular graphs(Lemma 6.9), Graphs and Combinatoric, 14(1998), 275–304, and H. Suzuki, Strongly closed subgraphs of a distance-regular graph with geometric girth five(Lemma 4.1), Kyushu Journal of Mathematics, 50(2)(1996), 371–384.)


Distance to x
0
……
i-1
i
i+1
x
……
u_1
u_3
u_1
u_2
u_5
u_3
u_4
0
i-1
i
i+1
x
……
u_2
u_3
u_1
u_4
u_5

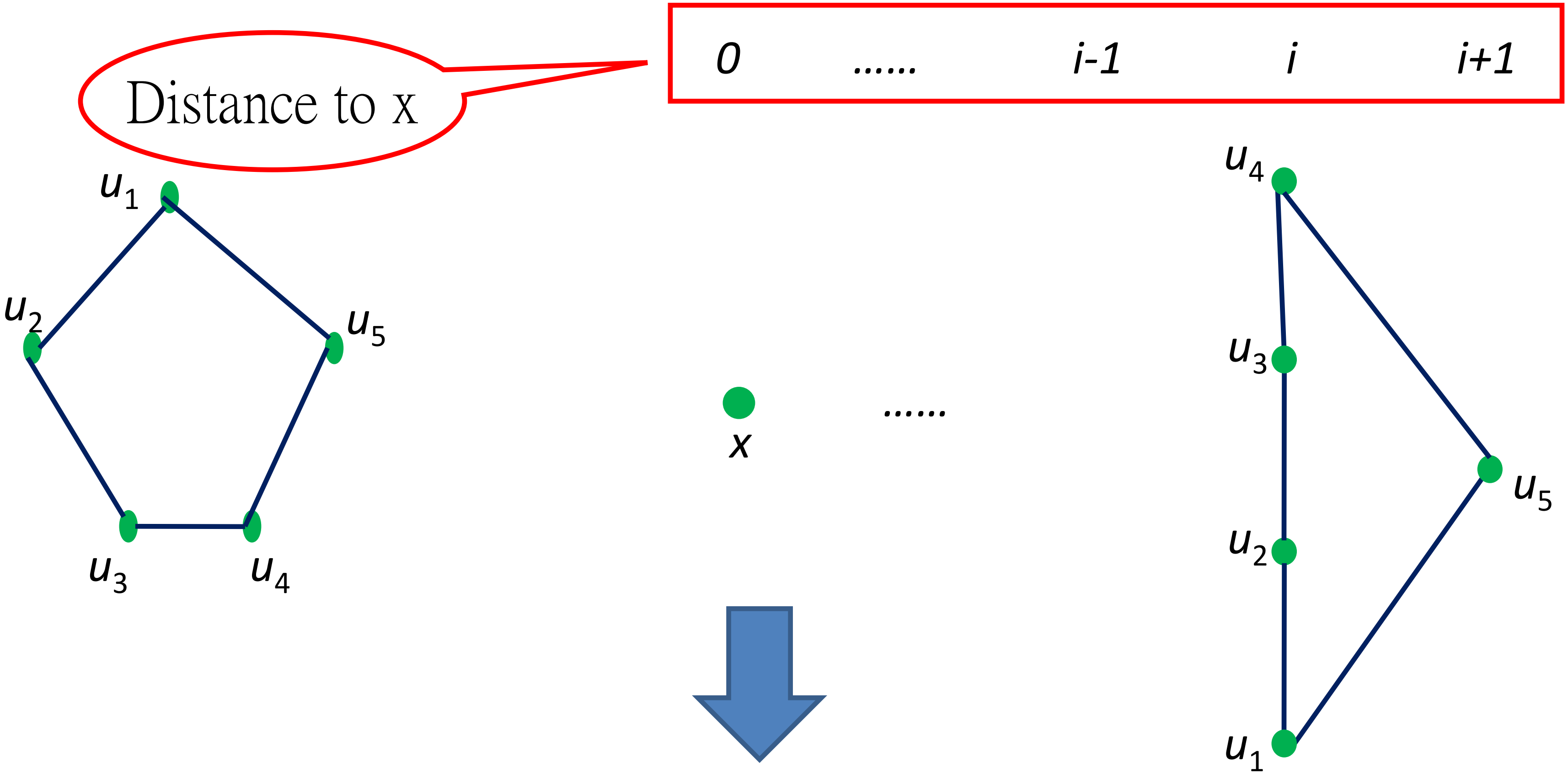


Does not exist !!!

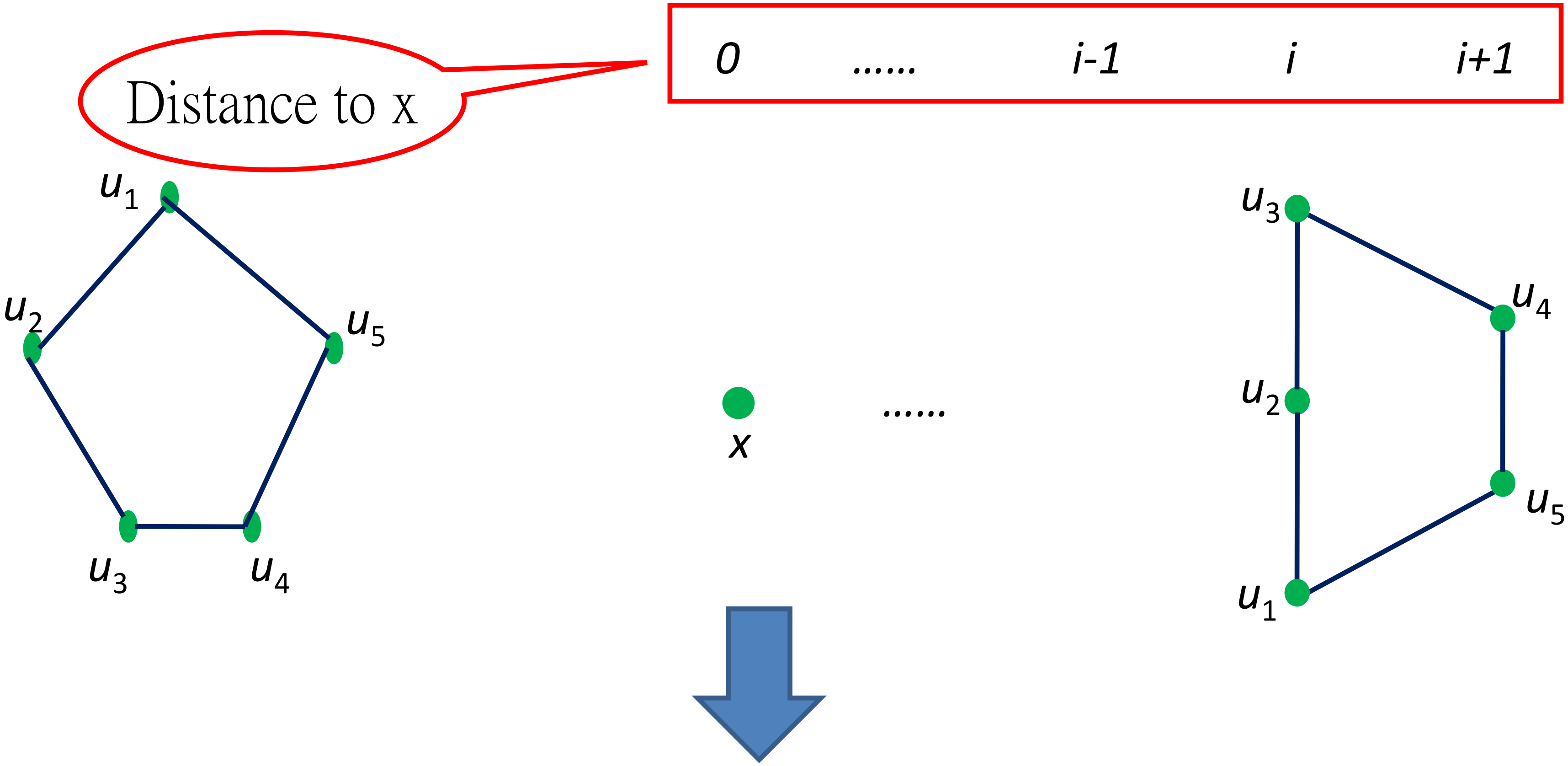
A. Hiraki's arguments

Lemma

Fix integers $1 \leq i \leq d \leq D-1$, *and suppose* Γ *does not contain parallelograms of any length up to* $d+1$. *Let* x *be a vertex of* Γ. *Then there is no pentagon of shape* $i, i, i, i, i+1$ *and no pentagon of shape* $i, i, i, i+1, i+1$ *with respect to* x *for* $1 \leq i \leq d$. □



Does not exist !!!



Does not exist !!!

The Construction

Definition

For any vertex $x \in X$ and any subset $\Pi \subseteq X$, define

$$[x, \Pi] := \{v \in X \mid \text{there exists } y' \in \Pi,$$

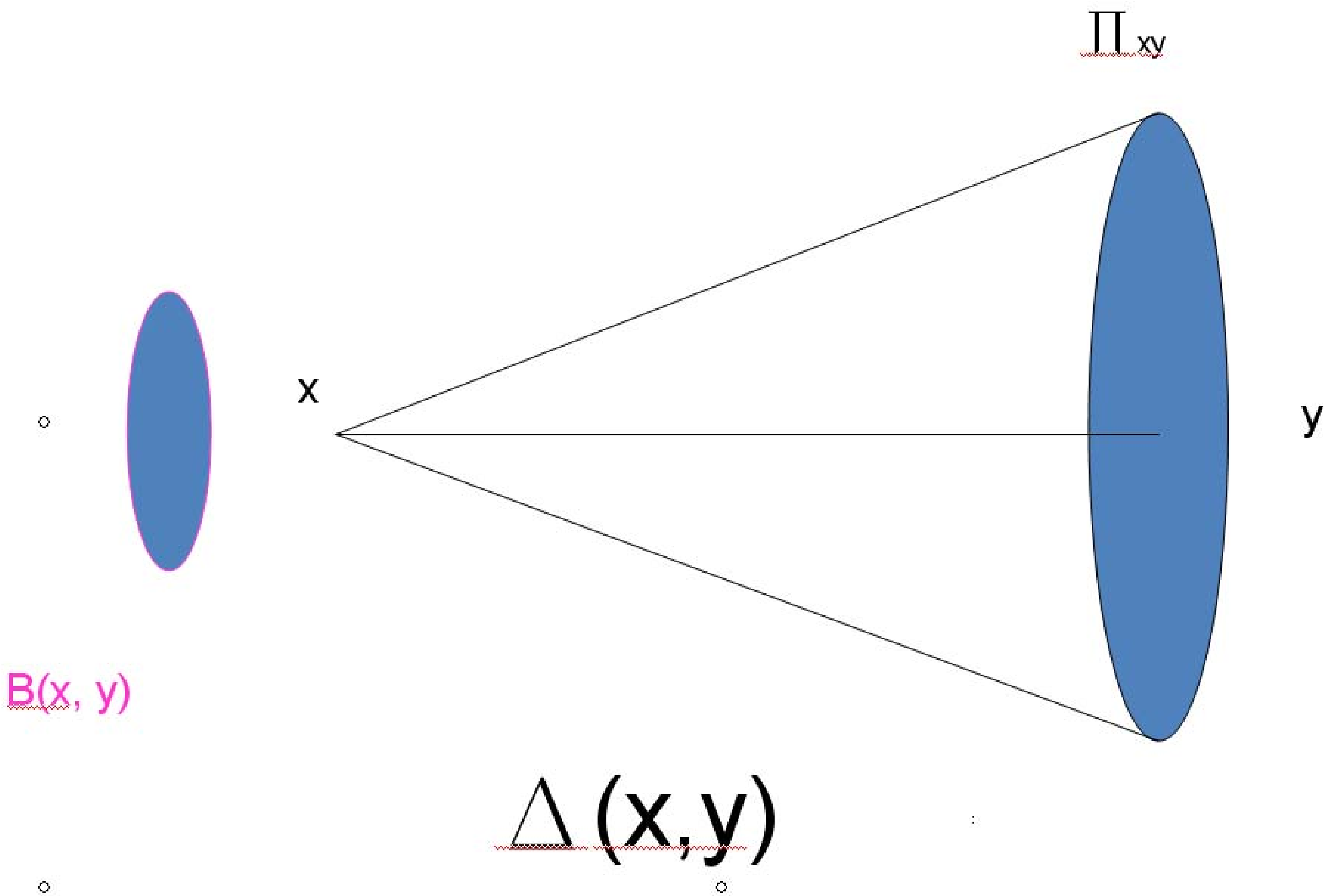
$$\text{such that the sequence } x, v, y' \text{ is geodetic } \}.$$

For any $x, y \in X$ with $\partial(x, y) = d$, set

$$\Pi_{xy} := \{y' \in \Gamma_d(x) \mid B(x, y) = B(x, y')\}$$

and

$$\Delta(x, y) = [x, \Pi_{xy}].$$


∏xy
x
y
B(x, y)
Δ (x,y)

It suffices to prove that

(W_d) $\Delta(x, y)$ is weak-geodetically closed with respect to x, and

(R_d) the subgraph induced on $\Delta(x, y)$ is regular with valency $a_d + c_d$.

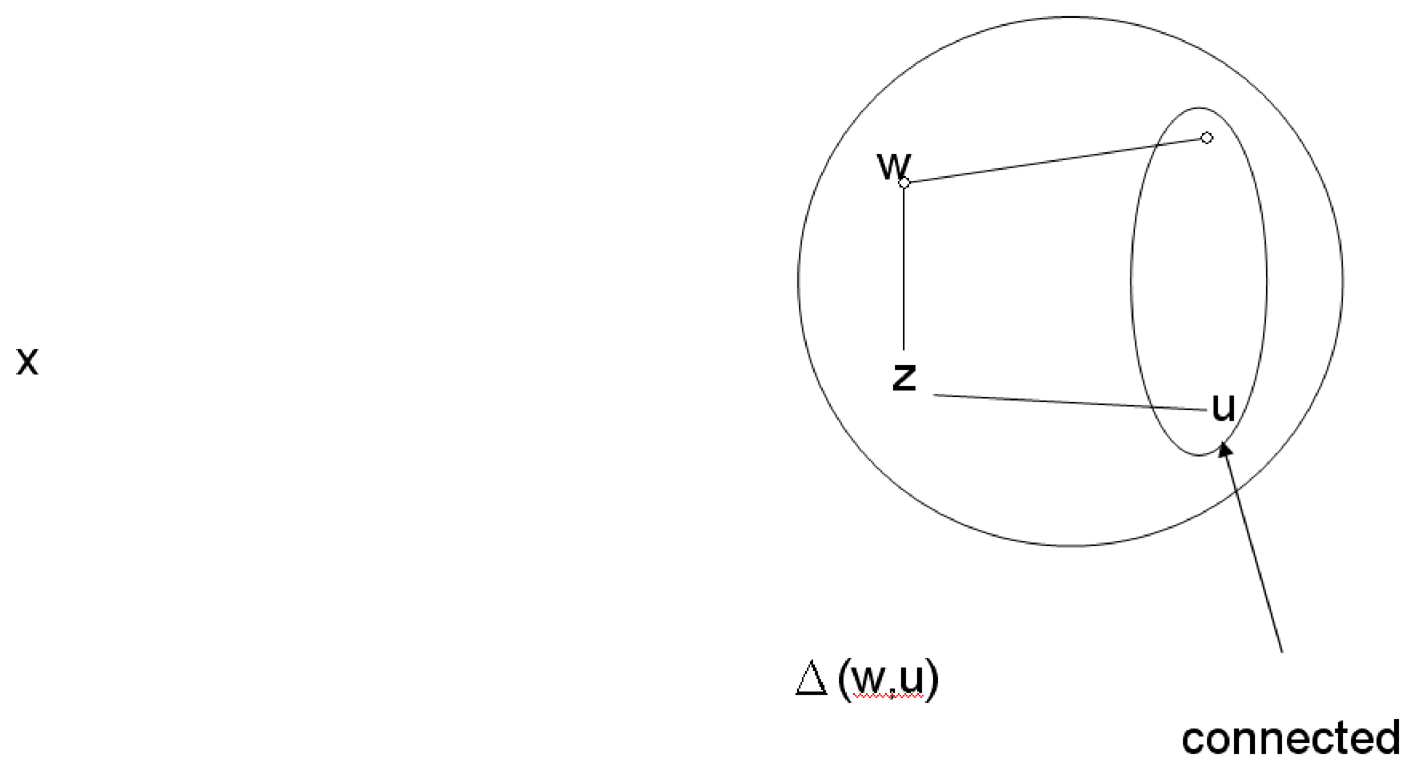
Lemma

Fix an integer $1 \leq d \leq D-1$, *and suppose* Γ *does not contain parallelograms of length up to* $d+1$. *Then for any two vertices* $z, z' \in X$ *such that* $\partial(x,z) \leq d$ *and* $z' \in A(z,x)$, *we have* $B(x,z) = B(x,z')$.

B(x, z)=B(x, z')

How to show $\Delta(x, y)$ is weak-geodetically closed with respect to x in the case $c_2>1$?

How to show Δ (x, y) is weak-geodetically closed with respect to x in the case $a_1>0$

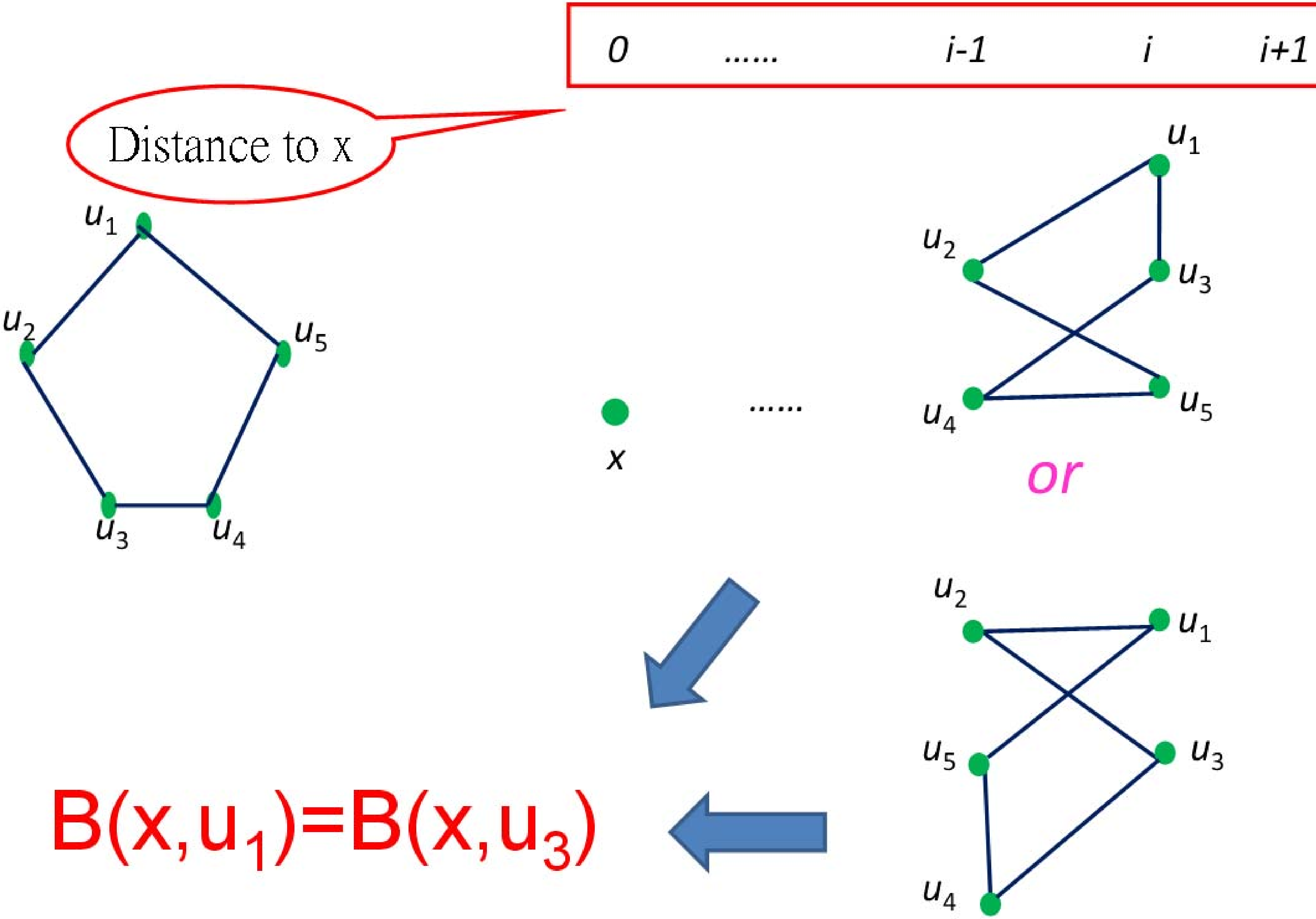


The case $a_1=0$ and $a_2>0$ is more complicate

The BB_d condition

Proposition

Fix integers $1 \leq i \leq d \leq D-1$, *and suppose* Γ *does not contain parallelograms of any length up to* $d+1$. *Let* x *be a vertex and* $u_1 u_2 u_3 u_4 u_5$ *be a pentagon of shape* $i, i-1, i, i-1, i$ *or of shape* $i, i-1, i, i-1, i-1$ *with respect to* x *for* $1 \leq i \leq d$ *for* $1 \leq i \leq d$. *Then* $B(x, u_1) = B(x, u_3)$.



$$B(x,u_1)=B(x,u_3)$$

Proposition

For any vertex $z \in \Delta(x, y) \cap \Gamma_i(x)$, where $1 \leq i \leq d$, we have the following (i), (ii).

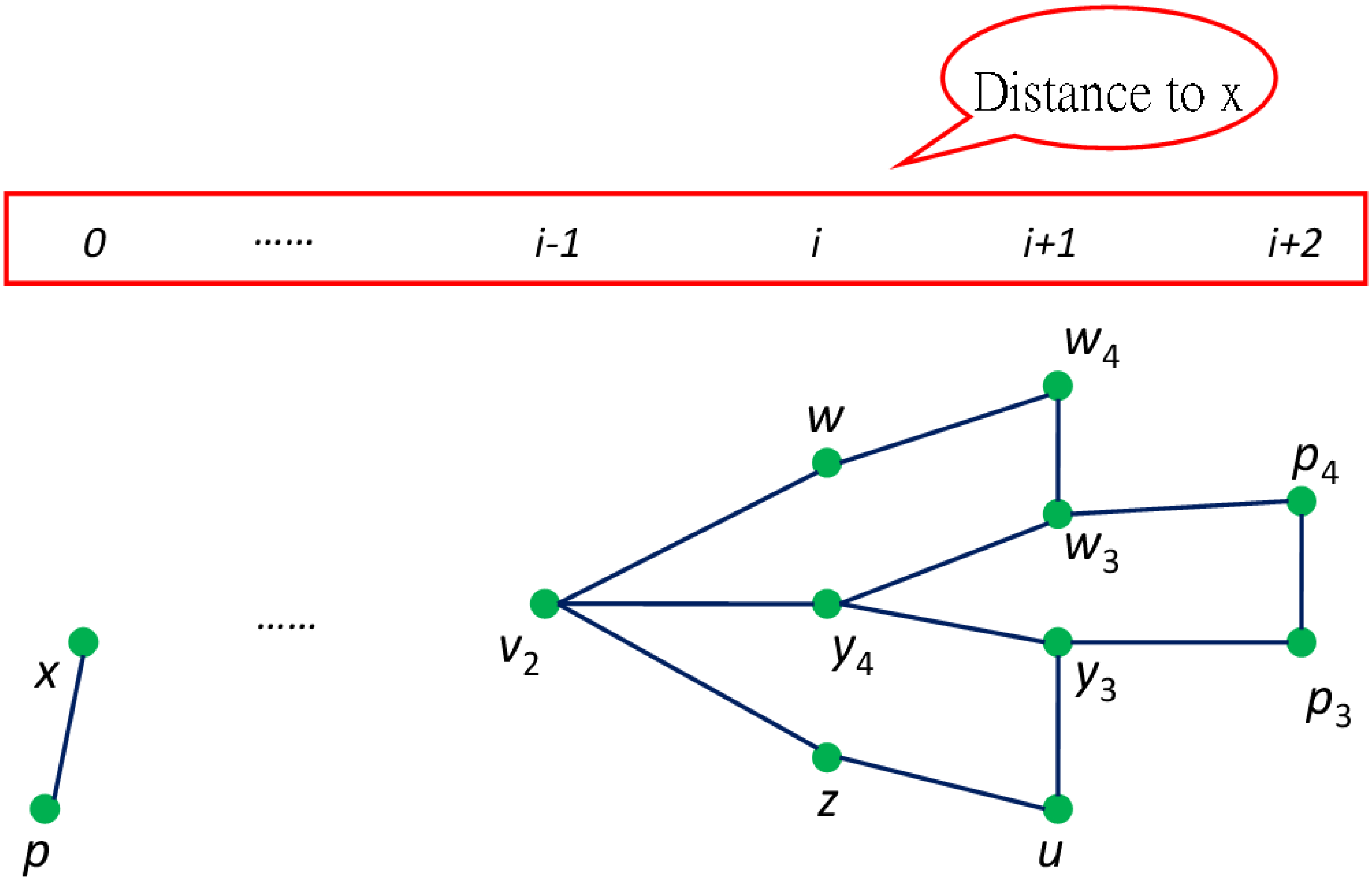
(i) $A(z, x) \subseteq \Delta(x, y)$.

(ii) *For any vertex $w \in \Gamma_i(x) \cap \Gamma_2(z)$ with $B(x, w) = B(x, z)$, we have $w \in \Delta(x, y)$.*

In particular the subgraph $\Delta(x, y)$ is weak-geodetically closed with respect to x.

$B(x, w)=B(x, z)$

A graph involved in the proof



Proposition

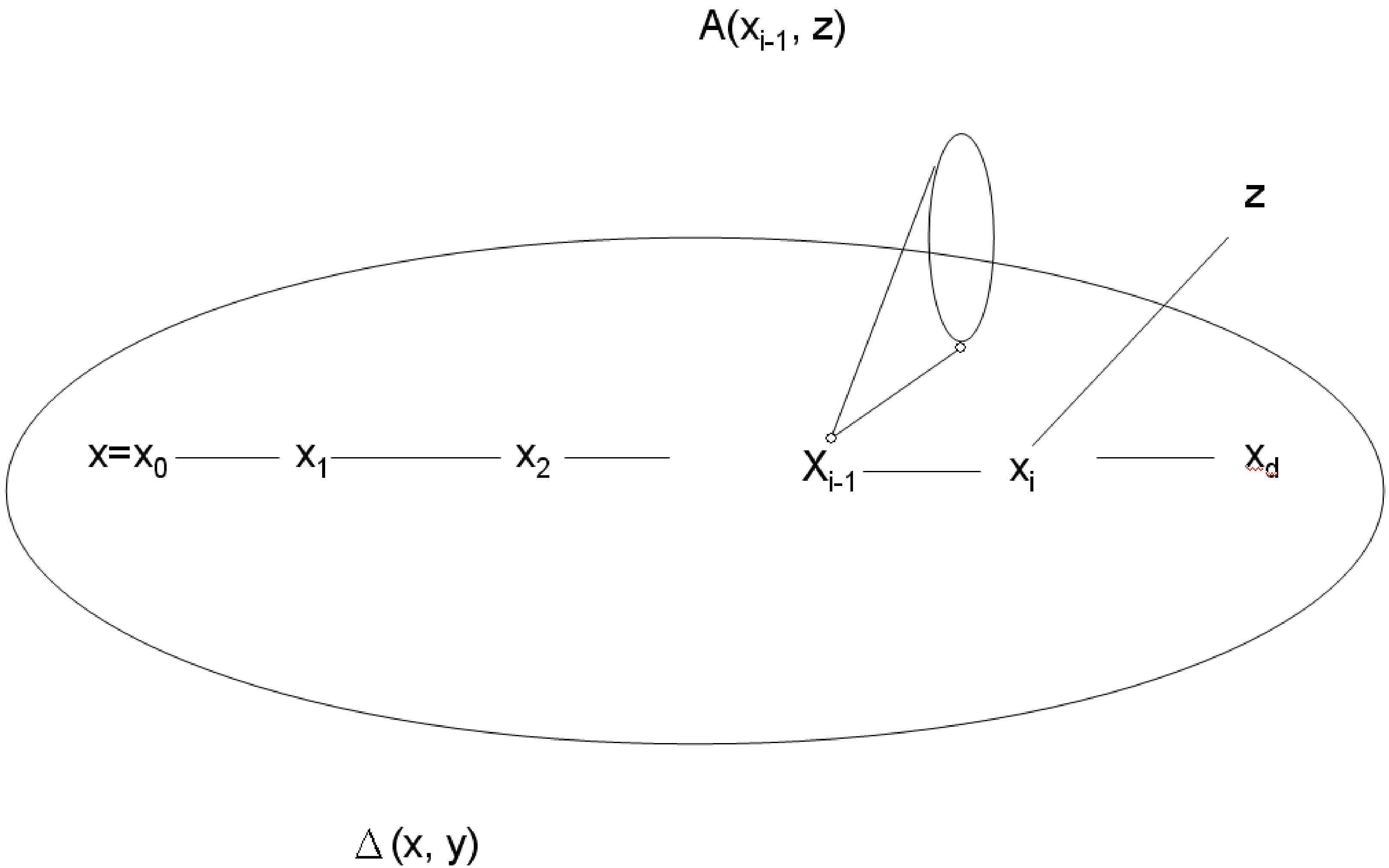
$\Delta(x, y)$ *is regular with valency* $a_d + c_d$.

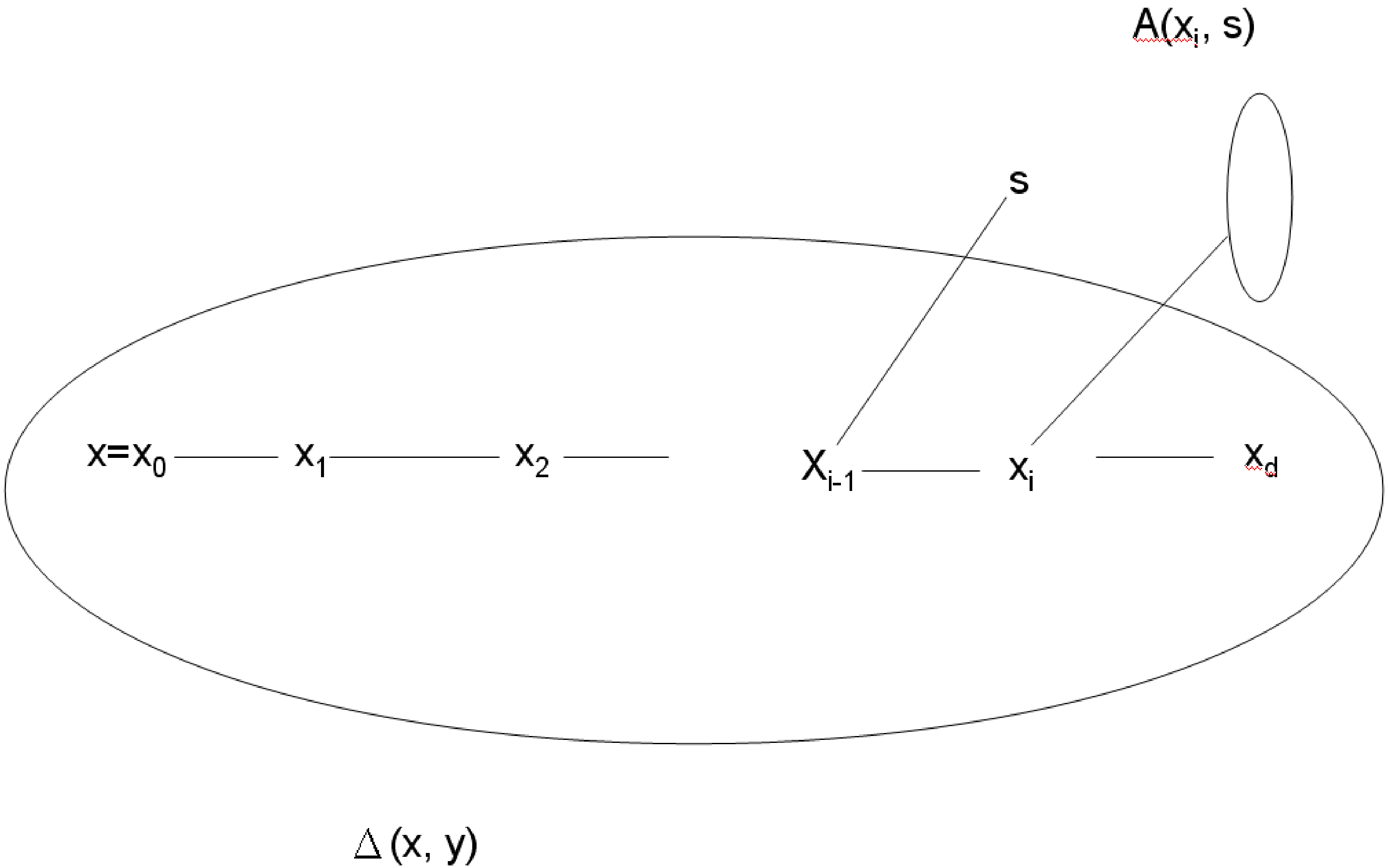
Idea of the proof

Since each vertex in $\Delta(x, y)$ appears in a sequence of vertices $x = x_0, x_1, \dots, x_d$ in Δ, where $\partial(x, x_j) = j$, $\partial(x_{j-1}, x_j) = 1$ for $1 \le j \le d$, and $x_d \in \Pi_{xy}$, it suffices to show

$$a_d + c_d = |\Gamma_1(x_0) \cap \Delta(x, y)| \ge |\Gamma_1(x_1) \cap \Delta(x, y)|$$

$$\ge |\Gamma_1(x_2) \cap \Delta(x, y)| \ge \cdots \ge |\Gamma_1(x_d) \cap \Delta(x, y)| = a_d + c_d.$$


A(x_{i-1}, z)
z
x=x_0
x_1
x_2
X_{i-1}
x_i
x_d
Δ (x, y)



A(x_i, s)
s
x=x_0
x_1
x_2
X_{i-1}
x_i
x_d
Δ (x, y)

From the above counting, we have

$$|\Gamma_1(x_{i-1}) \setminus \Delta(x,y)|a_2 \leq |\Gamma_1(x_i) \setminus \Delta(x,y)|a_2$$

for $1 \leq i \leq d$.

Application of Theorem to DRG with classical parameters

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Γ is said to have *classical parameters* (D, b, α, β) whenever the intersection numbers of Γ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix} \right) \qquad \text{for} \quad 0 \leq i \leq D,$$

$$b_i = \left(\begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \qquad \text{for} \quad 0 \leq i \leq D,$$

where

$$\begin{bmatrix} i \\ 1 \end{bmatrix} := 1 + b + b^2 + \cdots + b^{i-1}.$$

Theorem

Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) with $b < -1$ and $D \geq 4$. Suppose that Γ is D-bounded. Then

$$\beta = \alpha \frac{1 + b^D}{1 - b}. \tag{1}$$

(—, D-bounded distance-regular graphs (Theorem 4.2), European Journal of Combinatorics, 18(1997), 211–229.)

Ideal of the proof

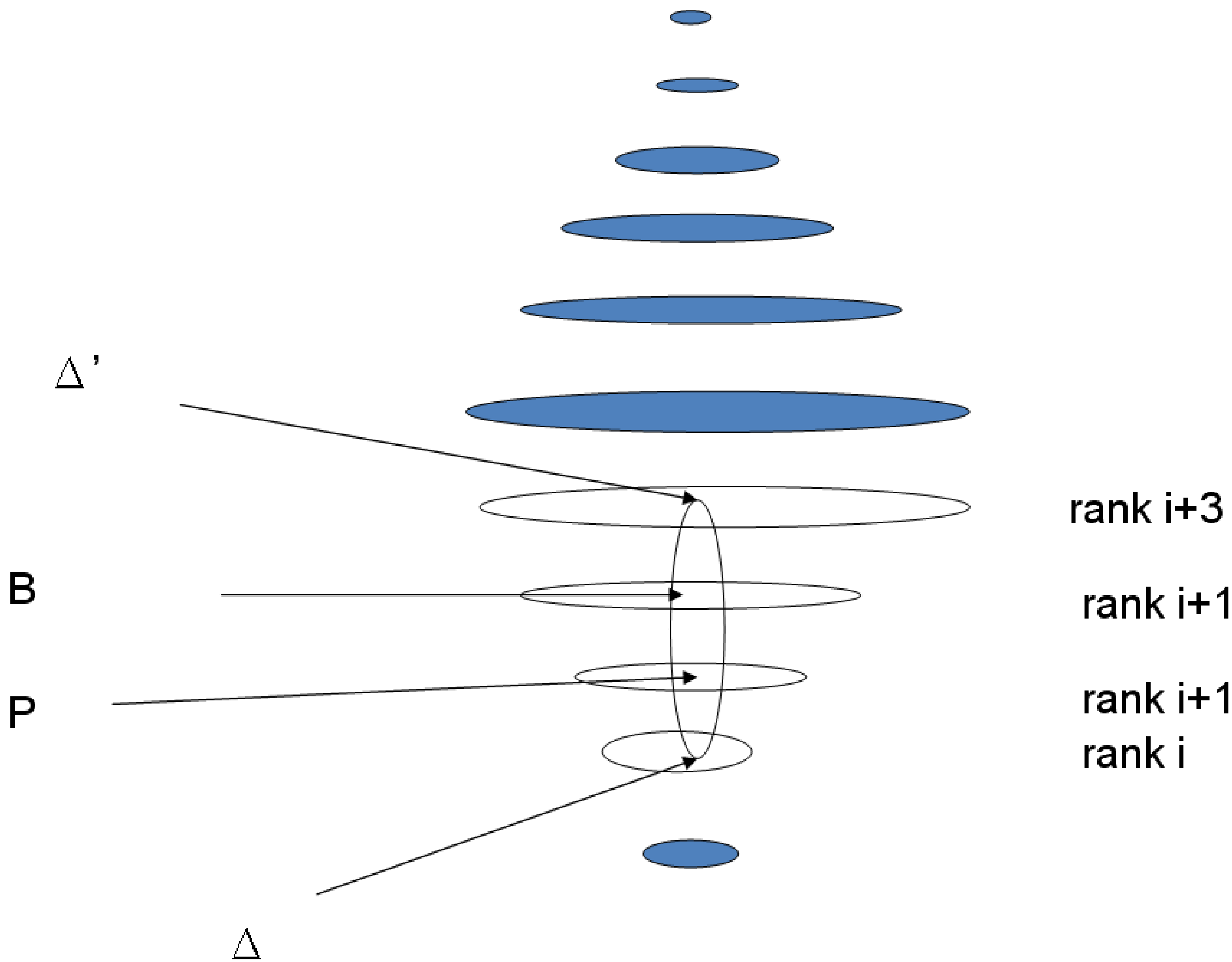
A regular weak-geodetically-closed of diameter i is called a subspace of rank i.

Fix a subspace Δ of dimension i and a subspace Δ' of rank i+3.

Let P (resp. B) be the set of subspaces of of rank i+1 (resp. i+2) containing Δ and contained in Δ'.

Then (P, B) is a 2-(v, k, 1) design to obtain Fisher's inequality.

The Fisher's equalities of two consecutive i become the desired identity. We need the assumption D>3 here.


Δ'
B
P
Δ
rank i+3
rank i+1
rank i+1
rank i

Corollary

Let Γ denote a distance-regular graph with classical parameters (D, b, α, β), $D \geq 4$ and $c_2 = 1$. Then $a_2 = a_1$ and $a_1 \neq 0$.

Conjecture

There is no distance-regular graph Γ *with classical parameters* (D, b, α, β), $D \geq 4$, *and* $c_2 = 1$.

Remark

(See [BCN, p. 194] The Triality graph ${}^3D_{4,2}(q)$ is a distance-regular graph with classical parameters $(3, -q, q/(1-q), q^2+q)$, $c_2 = 1$ and $a_1 = a_2 = q-1$. Hence the assumption $D \geq 4$ in Conjecture 0.2 is necessary. Note that The triality graph ${}^3D_{4,2}(q)$ is not 3-bounded since $b_1 = b_2$.

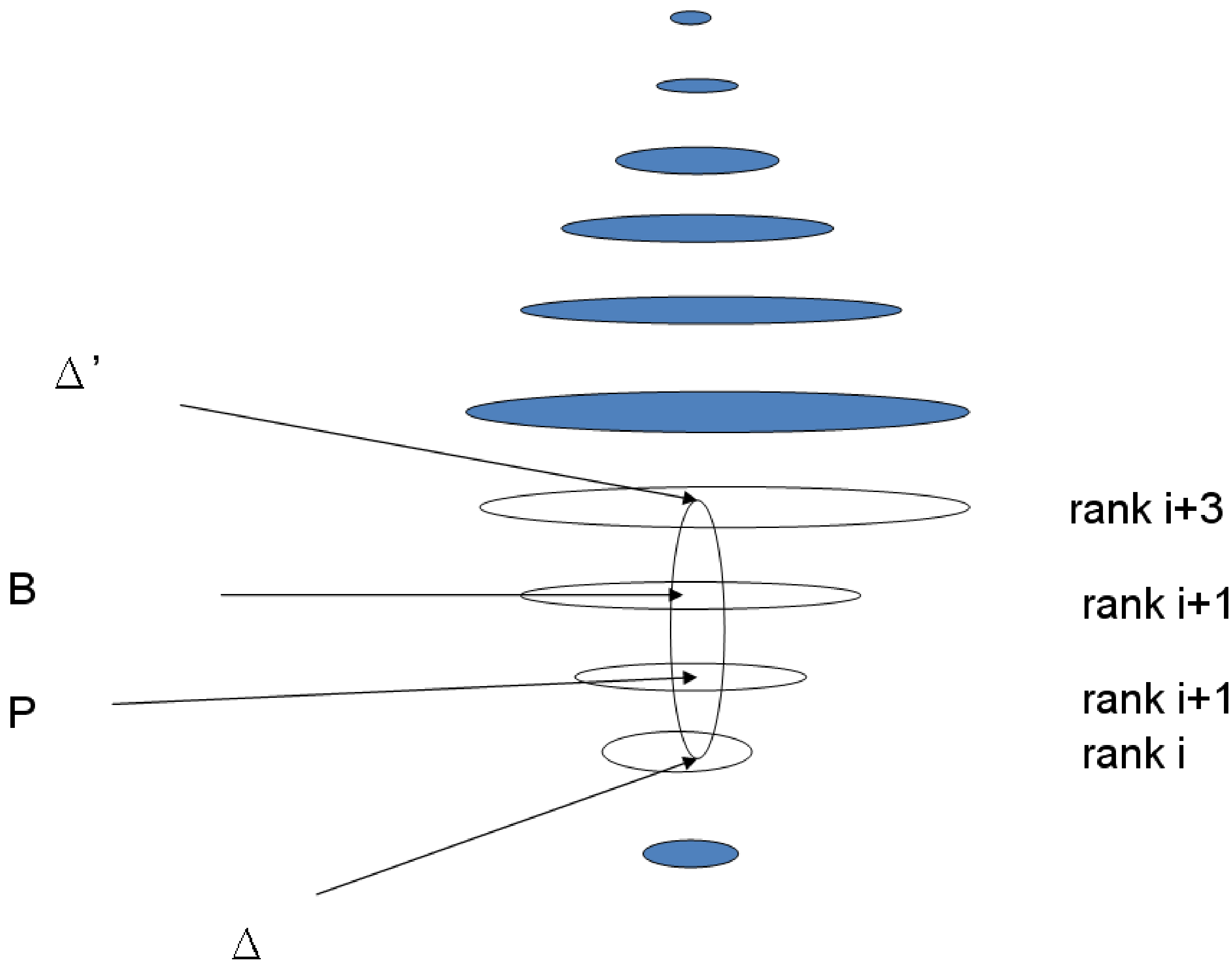
Another Application

Definition

A distance-regular graph Γ is said to have **geometric parameters** (D, b, α) whenever it has classical parameters (D, b, α, β), where $b \neq 1$ and

$$\beta = \alpha \frac{1 + b^D}{1 - b}.$$

(P, B) is a projective plane if DRG has geometric parameters



Lemma

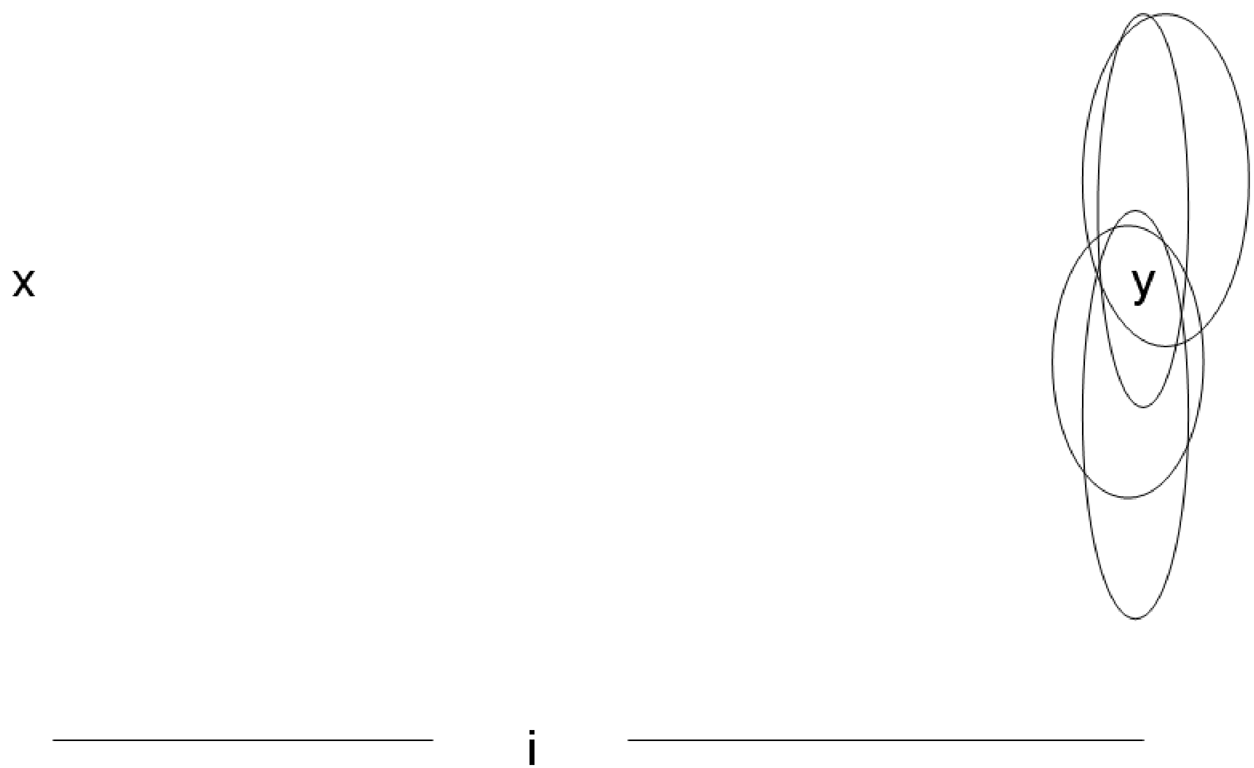
Let $\Gamma = (X, R)$ *denote a distance-regular graph with geometric parameters* (D, b, α) *and* $D \geq 4$. *Suppose* Γ *is* D*-bounded. Suppose* Γ *is not the dual polar graph* ${}^2A_{2D-1}(-b)$, *and* Γ *is not the Hermitian forms graph* $Her_{-b}(D)$. *Then* $\alpha = (b-1)/2$ *and* $-b$ *is a power of an odd prime.* □

(—, Classical distance-regular graphs of negative type, J. Combin. Theory Ser. B, 76(1999), 93–116.)

Idea of the proof

- A regular weak-geodetically closed subgraph of diameter 2 (resp. 1) is called a **plane** (resp. **line**).
- The **shape** of a plane with respect to a vertex x is the set of distances between the vertices in the plane and x.
- Fix two vertices x, y at distance i.
- The fact that the number of planes containing y of shape {i} with respect to x is nonnegative gives a useful equality.

Count the number of planes of shape {i} with respect to x

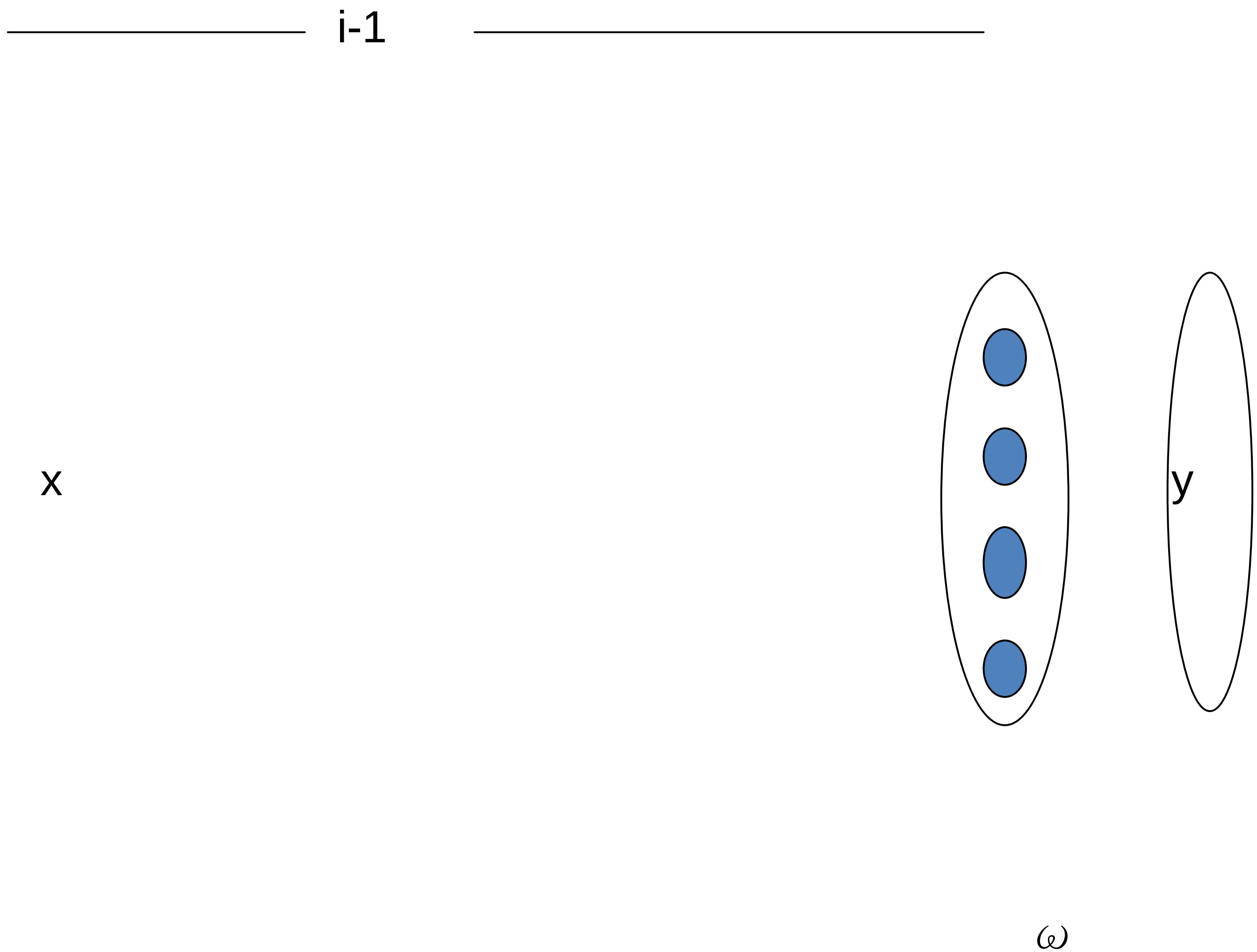


Ideal in the proof (conti.)

- Fix a plane ω containing y of shape $\{i\text{-}1, i\}$ with respect to x. Then $\omega \cap \Gamma_{i-1}(x)$ is a disjoint union of σ lines and $\sigma \neq b^2$.
- Use $(\sigma - b^2 - 1)/(\sigma - b^2) \geq 0$ to get another useful inequality.
- Two inequalities become an equality.

Ideal in the proof (conti.)

- Fix a plane ω containing y of shape $\{i\text{-}1, i\}$ with respect to x. Then $\omega \cap \Gamma_{i-1}(x)$ is a disjoint union of σ lines and $\sigma \neq b^2$.
- Use $(\sigma - b^2 - 1)/(\sigma - b^2) \geq 0$ to get another useful inequality.
- Two inequalities become an equality.


i-1
x
y
ω

Theorem

Let Γ denote a distance-regular graph with classical parameters (D, b, α, β) with $b < -1$, $D \geq 4$ and the intersection numbers $a_2 \neq 0$ and $b_1 > b_2$. Suppose Γ is not the dual polar graph ${}^2A_{2D-1}(-b)$, and Γ is not the Hermitian forms graph $Her_{-b}(D)$. Then $\alpha = (b-1)/2$, $\beta = -(1+b^D)/2$, and $-b$ is a power of an odd prime.

In the case a2>a1=0 and c_2>1, A. Hiraki can show that in the above theorem the assumption $D \geq 4$ can be loosen to $D \geq 3$, and b=-3 is the only remaining unknown case.

Thank You for Your Attention