On bipartite graphs analog of the Brualdi-Hoffman conjecture

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Spectral radius

When C is a real square matrix, the spectral radius $\rho(C)$ is defined as

$$\rho(C) := \max\{\ |\lambda|\ \ |\ \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ.

When C is nonnegative, $\rho(C)$ is known to be an eigenvalue of C.

A snapshot of our main method

The following is well known from the majorization-monotone property of spectral radii of nonnegative matrices :

$$\rho\left(\begin{array}{cc|c} 2 & 2 & 1 \\ 0 & 3 & 2 \\ \hline 1 & 2 & 1 \end{array}\right) \geq \rho\left(\begin{array}{cc|c} 2 & 1 & 1 \\ 0 & 3 & 1 \\ \hline 1 & 2 & 1 \end{array}\right) = \rho\left(\begin{array}{cc} 3 & 1 \\ 3 & 1 \end{array}\right) = 4.$$

Our main result implies

$$\rho\left(\begin{array}{cc|c} 2 & 2 & \color{red}{1} \\ 0 & 3 & \color{red}{2} \\ \hline 1 & 2 & \color{red}{1} \end{array}\right) \geq \rho\left(\begin{array}{cc|c} 2 & 1 & \color{red}{2} \\ 0 & 3 & \color{red}{2} \\ \hline 1 & 2 & \color{red}{1} \end{array}\right) = \rho\left(\begin{array}{cc} 3 & 2 \\ 3 & 1 \end{array}\right) = 2 + \sqrt{7}.$$

(One column exception is allowed in majorization-monotone property if the row-sums of two matrices are unchanged.)

Dual result

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right)$$

(same row-sums sequence)

$$\leq \rho\left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right) = \rho\begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix} \approx 18.69$$

Outline

0. Introduction

1. The non-complete bipartite graph with e edges which has the maximum spectral radius

2. The (non-complete) bipartite graph with e edges and bi-order p, q which has the maximum spectral radius

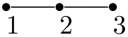
3. Spectral bounds of a nonnegative matrix

Notations

Let G denote a graph with $e = e(G)$ edges without isolated vertices. Let $A = A(G)$ be the adjacency matrix of G. The spectral radius $\rho(G)$ of G be the spectral radius of A.

Example

G



$$A = A(G) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$e = e(G) = 2, \ \rho(G) = \rho(A) = \sqrt{2}.$$

Spectral radii and graph invariants

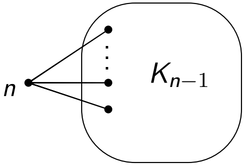
Let G be a graph of order n and size e with diameter D, minimum degree δ, maximum degree Δ, average degree $\bar{d}$, clique number ω and dominating number γ. The following are well-known in the spectral graph theory.

- $\delta \leq \bar{d} \leq \rho(G) \leq \Delta$
- $\omega \geq \frac{n}{n-\rho(G)}$
- $(n-1)^{\frac{1}{D}} \leq \rho(G) < \Delta - \frac{1}{nD}$
- If G is triangle-free, then $\rho(G) \leq \sqrt{e}$

Brualdi-Hoffman Conjecture (1976)

Conjecture

If $\binom{d}{2} < e \leq \binom{d+1}{2}$, the graph with the maximum spectral radius consists of the complete graph K_d to which a new vertex of degree $e - \binom{d}{2}$ is added, together with probably some isolated vertices.



Rowlinson proved this conjecture in 1988.

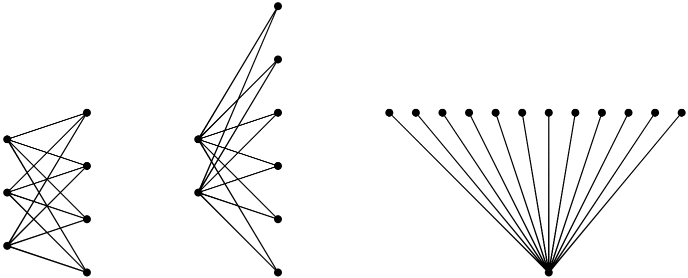
From now on, we assume G is bipartite with e edges.

A. Bhattacharya, S. Friedland, and U.N. Peled show the following.

Theorem (BFP 2008)

$$\rho(G) \leq \sqrt{e(G)}$$

with equality iff G is a complete bipartite graph with possible some isolated vertices. □



$e(G) = 12$

$\rho(G) = \sqrt{12}$

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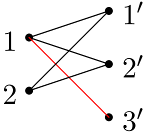
$K^{\pm}_{s,t}$

Define

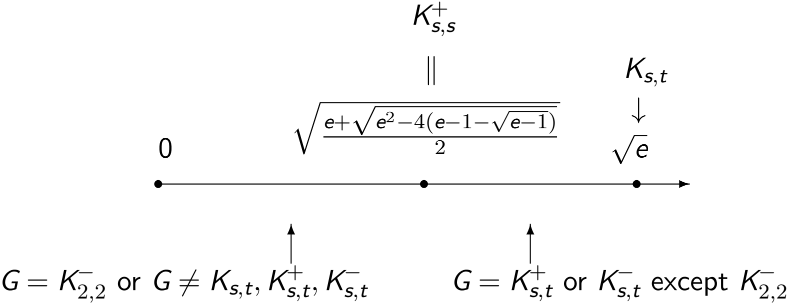
$$\begin{aligned} K^{-}_{s,t} &:= K_{s,t} - \{\overline{st}\}, \\ K^{+}_{s,t} &:= K_{s,t} + \{\overline{s(t+1)'}\} \qquad (2 \le s \le t). \end{aligned}$$

Example:



$$K^{-}_{2,3} = K^{+}_{2,2}$$

Value of $\rho(G)$ when e is fixed



Moreover we find

$$e = st - 1,\ s \searrow,\ t \nearrow \qquad \Rightarrow \qquad \rho(K^{-}_{s,t}) \nearrow,$$
$$e = st + 1,\ s \searrow,\ t \nearrow \qquad \Rightarrow \qquad \rho(K^{+}_{s,t}) \nearrow,$$

Extremal graphs

Theorem

If G has maximum spectral radius among bipartite non-complete graphs with e edges then

e	$(e-1, e+1)$	G
odd		$K_{2,t}^{-}$
even	*(prime,not prime)*	$K_{s,t}^{-}$ *with* $s \geq 2$ *the least*
even	*(not prime,prime)*	$K_{s,t}^{+}$ *with* $s \geq 2$ *the least*
even	*(not prime,not prime)* *neither primes case*	$K_{s,t}^{-}$ *with* $s \geq 2$ *the least or* $K_{s,t}^{+}$ *with* $s \geq 2$ *the least*
even	*(prime,prime)* *twin primes case*	*unknown (no* $K_{st}^{\pm}$ *with* $s \geq 2$*)*

□

Numerical comparisons of the neither primes case

In the case that $e \leq 100$ is even and neither $e-1$ nor $e+1$ is a prime, we determine which G of $K^-_{s,t}$ with $s \geq 2$ the least and $K^+_{s',t'}$ with $s' \geq 2$ the least has larger eigenvalue, where $e = st - 1 = s't' + 1$.

e	$\rho(K^-_{s,t})$	$\rho(K^+_{s',t'})$	winner
26	$\sqrt{13+3\sqrt{17}}$	$\sqrt{13+\sqrt{149}}$	$-$
34	$\sqrt{17+\sqrt{265}}$	$\sqrt{17+\sqrt{267}}$	$+$
50	$\sqrt{25+\sqrt{593}}$	$\sqrt{25+\sqrt{583}}$	$-$
56	$\sqrt{28+\sqrt{748}}$	$\sqrt{28+\sqrt{740}}$	$-$
64	$\sqrt{32+\sqrt{976}}$	$\sqrt{32+\sqrt{982}}$	$+$
76	$\sqrt{38+\sqrt{1384}}$	$\sqrt{38+\sqrt{1394}}$	$+$
86	$\sqrt{43+\sqrt{1813}}$	$\sqrt{43+\sqrt{1781}}$	$-$
92	$\sqrt{46+\sqrt{2096}}$	$\sqrt{46+\sqrt{2078}}$	$-$
94	$\sqrt{47+\sqrt{2137}}$	$\sqrt{47+\sqrt{2147}}$	$+$

A theorem for twin primes case

Let $\rho(e)$ denote the maximum $\rho(G)$ of a bipartite non-complete graph G with e edges.

Theorem

If $e \geq 4$ then $(e-1, e+1)$ is a pair of twin primes if and only if

$$\rho(e) < \sqrt{\frac{e + \sqrt{e^2 - 4(e - 1 - \sqrt{e-1})}}{2}}.$$

□

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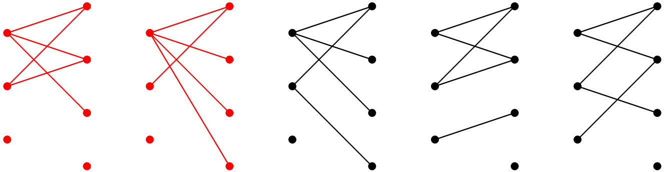
$\mathcal{K}(p, q, e)$ and $\mathcal{K}_0(p, q, e)$

Definition

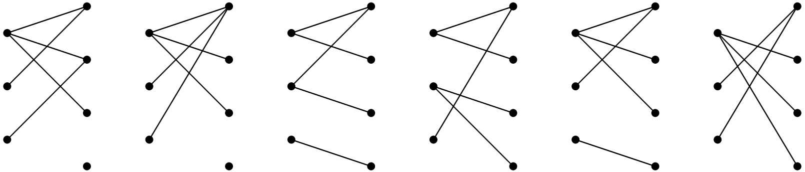
(i) $\mathcal{K}(p, q, e)$ is the family of subgraphs of $K_{p,q}$ with e edges without isolated vertices which are not complete bipartite graphs

(ii) $\mathcal{K}_0(p, q, e)$ is the subset of $\mathcal{K}(p, q, e)$ such that each graph in the subset is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

$\mathcal{K}(3,4,5)$, $\mathcal{K}_0(3,4,5)$ and $\rho(G)$



$\frac{\sqrt{10+2\sqrt{17}}}{2}$ $\quad$ $\frac{\sqrt{10+2\sqrt{13}}}{2}$ $\quad$ $\frac{\sqrt{10+2\sqrt{5}}}{2}$ $\quad$ 2 $\quad$ ≈ 1.8019



$\frac{\sqrt{6}+\sqrt{2}}{2}$ $\quad$ 2 $\quad$ $\sqrt{3}$ $\quad$ $\frac{1+\sqrt{5}}{2}$ $\quad$ $\sqrt{2+\sqrt{2}}$ $\quad$ $\sqrt{3}$

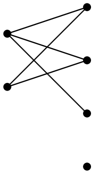
BFP Conjecture for $\mathcal{K}(p, q, e)$

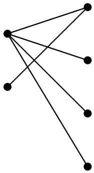
The following is a bipartite graphs analogue of Brualdi-Hoffman conjecture proposed by Bhattacharya, Friedland and Peled.

BFP Conjecture for $\mathcal{K}(p, q, e)$

If $G \in \mathcal{K}(p, q, e)$ such that $\rho(G) = \max_{H \in \mathcal{K}(p,q,e)} \rho(H)$ and $\mathcal{K}_0(p, q, e) \neq \emptyset$, then $G \in \mathcal{K}_0(p, q, e)$.

Example





$p = 2,\ q = 4,\ e = 5$

Some previous results

Theorem (Bhattacharya, Friedland and Peled 2008)

BFP Conjecture for $\mathcal{K}(p, q, e)$ holds for $e = st - 1$ for s, t satisfying $2 \leq s \leq p \leq t \leq t + (t-1)/(s-1)$. □

Theorem (Liu and Weng, 2015)

BFP Conjecture for $\mathcal{K}(p, q, e)$ holds for $e > pq - \min(p, q)$. □

Remark

The is no proper complete bipartite subgraph of $K_{p,q}$ with $e > pq - \min(p, q)$ edges.

A slight improvement

If $e \in \{st-1, st+1 \mid s \le p, t \le q\}$, then $K^-_{s,t} \in \mathcal{K}_0(p,q,e)$ or $K^+_{s,t} \in \mathcal{K}_0(p,q,e)$. The following theorem is an immediate consequence.

Theorem

BFP Conjecture for $\mathcal{K}(p,q,e)$ holds with

$$e \in \{st-1, st+1 \mid s \le p, t \le q\}.$$

□

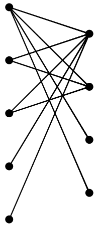
The graph G_D

For a sequence D of positive integers in nonincreasing order, one can define the bipartite graph G_D with bipartition $X = \{x_1, x_2, \ldots, x_p\}$, $Y = \{y_1, y_2, \ldots, y_{d_1}\}$ such that

$$E(G_D) = \{x_i y_j | 1 \leq i \leq p, 1 \leq j \leq d_i\}.$$

Example

For $D = (4, 2, 2, 1, 1)$ or $D = (5, 3, 1, 1)$, we have the isomorphic graph G_D.



$$G_{(4,2,2,1,1)} = G_{(5,3,1,1)}$$

Disproof of the BFP conjecture

Proposition

If $q > p + 2 \geq 5$ then BFP Conjecture for $\mathcal{K}(p, q, p(q-1))$ is false.

Proof.

With sequences

$$D_1 =(q, q-1, \ldots, q-1, q-2),$$
$$D_2 =(q, q, \ldots, q, q-p),$$

$G_{D_1}, G_{D_2} \in \mathcal{K}(p, q, p(q-1))$ and $\mathcal{K}_0(p, q, p(q-1) = \{G_{D_2}\}$. By direct computation, $\rho(G_{D_2}) < \rho(G_{D_1})$. □

$\mathcal{C}(p, q, e)$

From now on the complete bipartite graphs will be included in our consideration.

Definition

(i) $\mathcal{C}(p, q, e)$ is the family of subgraphs of $K_{p,q}$ with e edges without isolated vertices.

(ii) $\mathcal{C}_0(p, q, e)$ is the subset of $\mathcal{C}(p, q, e)$ such that each graph in the subset is a complete bipartite graph or a graph obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

Weak BFP Conjecture

We propose the following conjecture.

Weak BFP Conjecture for $\mathcal{C}(p, q, e)$

If $G \in \mathcal{C}(p, q, e)$ such that $\rho(G) = \max_{H \in \mathcal{C}(p,q,e)} \rho(H)$, then $G \in \mathcal{C}_0(p, q, e)$.

$e \geq pq - \max(p, q)$ or $p \leq 5$

We have the following two theorems.

Theorem

If $e \geq pq - \max(p, q)$ then the weak BFP Conjecture for $\mathcal{C}(p, q, e)$ is true. □

Theorem

If $\min(p, q) \leq 5$ then the weak BFP conjecture for $\mathcal{C}(p, q, e)$ is true. □

The proofs of the above two Theorems employ some new sharp upper bounds of the spectral radii of nonnegative matrices which will be the last part of my talk.

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Motivation

A bipartite graph G has adjacency matrix of the block form

$$A(G) = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}.$$

Then

$$A^2 = \begin{pmatrix} BB^T & O \\ O & B^TB \end{pmatrix}.$$

Since BB^T and B^TB have the same spectral radius,

$$\rho^2(G) = \rho(BB^T) = \rho(B^TB).$$

Because BB^T is no longer a binary matrix, we need spectral theory for general nonnegative matrices C.

Motivation

Let C and C' be two $n \times n$ nonnegative matrix. It is well-known as a consequence of Perron-Frobenius Theorem that

$$C \leq C' \quad \Rightarrow \quad \rho(C) \leq \rho(C').$$

Moreover if C is irreducible then $\rho(C) = \rho(C')$ if and only if $C = C'$.

We might expect to find another matrix C' such there are many C related to C' in some way and $\rho(C) \leq \rho(C')$. Moreover we expect the matrix C with $\rho(C) = \rho(C')$ is not unique.

Matrix notation

For a matrix M and a subset α of the set of row indices and a subset β of the set of column indices, we use $M[\alpha|\beta]$ to denote the submatrix of M which restricts the positions in $\alpha \times \beta$.

Example

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad M[[4]|[3]] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \; M[\{4\}|[4]] = (1, 1, 0, 0),$$

where $[n] := \{1, 2, \dots, n\}$.

Rooted matrix

An $m \times n$ matrix $C' = (c'_{ij})$ is rooted if

$$c'_{ij} \geq c'_{nj} \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq j \leq n-1, \text{ and}$$

$$r'_i := \sum_{j=1}^{n} c'_{ij} \geq r'_n := \sum_{j=1}^{n} c'_{nj} \qquad \text{for } 1 \leq i \leq m-1.$$

Example

The matrix

$$C' = \begin{pmatrix} 6 & 6 & -1 \\ 8 & 2 & 0 \\ 5 & 2 & 0 \end{pmatrix}$$

with row-sum vector $(r'_1, r'_2, r'_3)^T = (11, 10, 7)^T$, which is a rooted vector.

$\rho_r(C')$

Remark

As a rooted matrix C' is not always nonnegative, $\rho(C')$ is not necessary to be the largest real eigenvalue of C'. Let $\rho_r(C')$ denote the largest real eigenvalue of C' (Its existence is proved).

A comment on rooted matrix

Remark

C' is nonnegative $\Rightarrow$ $\begin{pmatrix} C' & 0 \\ u & a \end{pmatrix}$ is rooted and $\rho(C') = \rho \begin{pmatrix} C' & 0 \\ u & a \end{pmatrix}$

for suitable chosen of row vector $u \geq 0$ and scalar $a \leq 0$ to have 0 row-sum in the last row.

Construct C' from C

From an $n \times n$ matrix $C = (c_{ij})$, we construct another $n \times n$ matrix $C' = (c'_{ij})$ that satisfies

(i) $C[[n]|[n-1]] \leq C'[[n]|[n-1]]$;

(ii) $r_i := \sum_{j=1}^{n} c_{ij} \leq r'_i := \sum_{j=1}^{n} c'_{ij} \qquad$ for $1 \leq i \leq n$;

(iii) $C' + kI$ is rooted for some k;

(iv) $C'[\{n\}|[n-1]] > 0$.

Example

$$C = \begin{pmatrix} 3 & 6 & 2 \\ 8 & 1 & 1 \\ 5 & 5 & 0 \end{pmatrix}, \qquad C' = \begin{pmatrix} 6 & 6 & -1 \\ 8 & 2 & 0 \\ 5 & 5 & 0 \end{pmatrix}$$

A restricted form of our main method

Theorem

With the notation from the last page, we have

$$\rho(C) \leq \rho_r(C').$$

□

The set K

To study the case of equality $\rho(C) = \rho(C')$ of the theorem in the last page, we need information of the eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T$ (known to be positive and rooted) of C' for $\rho_r(C')$ and the set

$$K = \{j \mid v'_j > v'_n\}.$$

An easier way to find K

$$\text{Let } K_1 = \{i \mid r'_i > r'_n\}, \quad \text{and when } K_t \text{ is defined, let}$$
$$K_{t+1} = \{i \notin \bigcup_{s \le t} K_s \mid c'_{ij} > c'_{nj} \text{ for some } j \in \textstyle\bigcup_{s \le t} K_s\}.$$

Lemma

If $r'_i > 0$ for $1 \le i \le n-1$ then

$$\begin{cases} K = \emptyset, & \textit{if } K_1 = \emptyset; \\ K = \bigcup_{s=1}^{\infty} K_s & \textit{otherwise.} \end{cases}$$

□

The equality part of the theorem

Theorem

With the notation in the last few pages, if C is irreducible and $r'_i > 0$ for $1 \leq i \leq n-1$, then $\rho(C) = \rho_r(C')$ if and only if

$$\begin{aligned} r_i &= r'_i \quad \text{for } 1 \leq i \leq n \\ c'_{ij} &= c_{ij} \quad \text{for } 1 \leq i \leq n \text{ and } j \in K. \end{aligned}$$

□

(c_{ij} is free if $j \notin K$.)

A non-example holds

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right) \leq \rho\left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right)$$

Although the matrix on the right violates some pieces of the assumptions in C', the above inequality still holds.

The matrix C' after equitable quotient matters

$$\rho \left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & \color{red}{0} \\ 4 & 2 & 1 & 4 & 2 & 6 & \color{red}{4} \\ 2 & 3 & 1 & 4 & 1 & 8 & \color{red}{3} \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & \color{red}{4} \\ 5 & 6 & 1 & 1 & 0 & 3 & \color{red}{3} \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & \color{red}{0} \\ 2 & 2 & 0 & 2 & 1 & 1 & \color{red}{4} \end{array}\right) \qquad \text{(same row-sums sequence)}$$

$$\leq \rho \left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & \color{red}{-1} \\ 4 & 2 & 1 & 4 & 2 & 6 & \color{red}{5} \\ 2 & 3 & 2 & 4 & 2 & 8 & \color{red}{3} \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & \color{red}{3} \\ 5 & 6 & 1 & 1 & 1 & 3 & \color{red}{3} \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & \color{red}{-1} \\ 2 & 2 & 0 & 2 & 2 & 1 & \color{red}{4} \end{array}\right) = \rho \begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix} \approx 18.69$$

Applications

In the following few pages, we shall provide some applications of the inequality $\rho(C) \leq \rho_r(C')$.

The matrices C attaining the equality can be characterized, but for simplicity, we omit the discussion here.

Realization a result of Xing Duan and Bo Zhou

Theorem

Let $C = (c_{ij})$ be a nonnegative $n \times n$ matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$ and $d := \max_i c_{ii}$, $f := \max_{i \neq j} c_{ij}$. Then for $1 \leq \ell \leq n$,

$$\rho(C) \leq \frac{r_\ell + d - f + \sqrt{(r_\ell - d + f)^2 + 4f\sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2}$$

Proof.

$$C' = \begin{pmatrix} d & f & \cdots & f & r_1 - (\ell-2)f - d \\ f & d & & f & r_2 - (\ell-2)f - d \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{\ell-1} - (\ell-2)f - d \\ f & f & \cdots & f & r_\ell - (\ell-1)f \end{pmatrix}_{\ell \times \ell}.$$

□

A little generalization

Theorem

Let $C = (c_{ij})$ *be a nonnegative* $n \times n$ *matrix with row-sums* $r_1 \geq r_2 \geq \cdots \geq r_n$ *and* $d \geq \max_{1 \leq i \leq \ell-1} c_{ii}$, $f \geq \max_{1 \leq i \neq j \leq \ell-1} c_{ij}$. *Then for* $1 \leq \ell \leq n$,

$$\rho(C) \leq \frac{r_\ell + d - f + \sqrt{(r_\ell - d + f)^2 + 4f \sum_{i=1}^{\ell-1} (r_i - r_\ell)}}{2}$$

Proof.

$$C' = \begin{pmatrix} d & f & \cdots & f & r_1 - (\ell-2)f - d \\ f & d & & f & r_2 - (\ell-2)f - d \\ \vdots & & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_{\ell-1} - (\ell-2)f - d \\ f & f & \cdots & f & r_\ell - (\ell-1)f \end{pmatrix}_{\ell \times \ell} .$$

□

A theorem of Richard Stanley in 1987

Theorem

Let $C = (c_{ij})$ be an $n \times n$ symmetric (0,1) matrix with zero trace. Let the number of 1's of C be $2e$. Then

$$\rho(C) \leq \frac{-1+\sqrt{1+8e}}{2}.$$

Proof.

Use $2e = \sum_{i=1}^{n} r_n$ and

$$C' = \begin{pmatrix} 0 & 1 & \cdots & 1 & r_1 - (n-1) \\ 1 & 0 & \ddots & 1 & r_2 - (n-1) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & r_n - (n-1) \\ 1 & 1 & \cdots & 1 & -n \end{pmatrix}.$$

□

A generalization to nonnegative matrices

Theorem

Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix. Let m be the sum of entries, and $d \geq \max_i c_{ii}$, $f \geq \max_{i \neq j} c_{ij}$. Then

$$\rho(C) \leq \frac{d - f + \sqrt{(d-f)^2 + 4fm}}{2}.$$

Proof.

Use $m = \sum_{i=1}^{n} r_n = n(n-1)f + nd$ and

$$C' = \begin{pmatrix} d & f & \cdots & f & r_1 - d - (n-1)f \\ f & d & \ddots & f & r_2 - d - (n-1)f \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ f & f & \cdots & d & r_n - d - (n-1)f \\ f & f & \cdots & f & -nf \end{pmatrix}.$$

□

Realization a result of Csikvári in 2009

Theorem

Assume that the set $\{v_1, v_2, \ldots, v_k\}$ *forms a clique in the graph* G *and* $V(G) \setminus K = \{v_{k+1}, \ldots, v_n\}$ *forms an independent set. Let* e' *be the number of edges between* K *and* $V(G) \setminus K$. *Then*

$$\rho(G) \leq \frac{k-1+\sqrt{(k-1)^2+4e'}}{2}.$$

Proof.

Use $e' = \sum_{i=1}^{k} r_i$ and

$$C' = \begin{pmatrix} 0 & 1 & \cdots & 1 & r_1-(k-1) \\ 1 & 0 & \ddots & 1 & r_2-(k-1) \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & r_k-(k-1) \\ 1 & 1 & \cdots & 1 & 0 \end{pmatrix}.$$

$\rho(G) \leq \rho(G_{D(G)})$

To illustrate how our method is applied to bipartite graph, we need the following theorem of A. Bhattacharya, S. Friedland, and U.N. Peled in 2008.

Theorem

If a bipartite graph G has degree sequence $D = D(G)$ of one part then $\rho(G) \leq \rho(G_D)$ with equality if and only if $G = G_D$ (up to isomorphism). □

The spectral radius of G_D

The bipartite graph G_D has adjacency matrix of the block form

$A(G) = \begin{pmatrix} 0 & B(D) \\ B(D)^T & 0 \end{pmatrix}$. Then $A^2 = \begin{pmatrix} B(D)B(D)^T & O \\ O & B(D)^TB(D) \end{pmatrix}$,

and

$$C := B(D)B(D)^T = \begin{pmatrix} d_1 & d_2 & d_3 & & d_p \\ d_2 & d_2 & d_3 & & d_p \\ d_3 & d_3 & d_3 & & d_p \\ & & & \ddots & \\ d_p & d_p & d_p & & d_p \end{pmatrix}.$$

Since $B(D)B(D)^T$ and $B(D)^TB(D)$ have the same spectral radius,

$$\rho^2(G_D) = \rho(A^2) = \rho(C).$$

A proof of the next theorem

For $D = (d_1, d_2, \ldots, d_p)$ in nonincreasing order,

$$C = \begin{pmatrix} d_1 & d_2 & d_3 & & d_p \\ d_2 & d_2 & d_3 & & d_p \\ d_3 & d_3 & d_3 & & d_p \\ & & & \ddots & \\ d_p & d_p & d_p & & d_p \end{pmatrix}$$

and fix $1 \leq \ell \leq p$, we will choose

$$C' = \begin{pmatrix} d_1 & d_1 & \cdots & d_1 & r_1 - (\ell - 1)d_1 \\ d_2 & d_2 & \cdots & d_2 & r_2 - (\ell - 1)d_2 \\ d_3 & d_3 & \cdots & d_3 & r_3 - (\ell - 1)d_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_\ell & d_\ell & \cdots & d_\ell & r_\ell - (\ell - 1)d_\ell \end{pmatrix},$$

where $r_i = (i-1)d_i + \sum_{k=i}^{p} d_k$ is the i-th row-sum of C to obtain the theorem in the next page.

A theorem for bipartite graphs

Theorem

Let G be a bipartite graph and $D = (d_1, d_2, \ldots, d_p)$ be the degree sequence of one part of G in decreasing order. Then for $1 \leq \ell \leq p$,

$$\rho(G) \leq \sqrt{\frac{r_1 + \sqrt{(2r_\ell - r_1)^2 + 4d_\ell \sum_{i=1}^{\ell}(r_i - r_\ell)}}{2}}.$$

□

The above theorem is the main tool for our proof of the weak BFP Conjecture for $C(p, q, e)$ with $e \geq pq - \max(p, q)$ or $\min(p, q) \leq 5$.

A new lower bound

Our method also has dual version.

Theorem

Let $C = (c_{ij})$ be an $n \times n$ nonnegative matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq t < n$, let $d = \max_{t<i\leq n} c_{ii}$ and $f = \max_{1\leq i\leq n, t<j\leq n, i\neq j} c_{ij}$. Assume that $0 < r_n - (n-t-1)f - d$. Then

$$\frac{r_t - f + d + \sqrt{(r_t - (2n-2t-1)f - d)^2 + 4(n-t)(fr_n - (n-t-1)f - d)}}{2}$$

is a lower bound of $\rho(C)$.

Proof.

$$C' = \begin{pmatrix} r_t - (n-t)f & (n-t)f \\ r_n - (n-t-1)f - d & (n-t-1)f + d \end{pmatrix}.$$

□

A general form of our main method

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that

(i) $PCQ \leq PC'Q$;

(ii) C' *has an eigenvector* Qu *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T \geq 0$ *and* $\lambda' \in \mathbb{R}$;

(iii) C *has a left eigenvector* $v^T P$ *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n) \geq 0$ *and* $\lambda \in \mathbb{R}$; *and*

(iv) $v^T PQu > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

□

Quick realization

To realize the theorem in the last page, we might investigate its special case $P = Q = I$.

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ matrices. Assume that

(i) $C \leq C'$;

(ii) C' *has an eigenvector* u *for* λ' *for some nonnegative column vector* $u = (u_1, u_2, \ldots, u_n)^T \geq 0$ *and* $\lambda' \in \mathbb{R}$;

(iii) C *has a left eigenvector* v^T *for* λ *for some nonnegative row vector* $v^T = (v_1, v_2, \ldots, v_n) \geq 0$ *and* $\lambda \in \mathbb{R}$; *and*

(iv) $v^T u > 0$.

Then $\lambda \leq \lambda'$. *Moreover,* $\lambda = \lambda'$ *if and only if*

$$(C')_{ij} = (C)_{ij} \qquad \text{for } 1 \leq i, j \leq n \text{ with } v_i \neq 0 \text{ and } u_j \neq 0.$$

□

As simple as majorization-monotone property

Theorem

Let $C = (c_{ij})$, $C' = (c'_{ij})$ be $n \times n$ matrices. Assume that

(i) $0 \leq C \leq C'$;

(ii) ~~C' has an eigenvector u for λ' for some nonnegative column vector $u = (u_1, u_2, \ldots, u_n)^T \geq 0$ and $\lambda' \in \mathbb{R}$;~~

(iii) ~~C has a left eigenvector v^T for λ for some nonnegative row vector $v^T = (v_1, v_2, \ldots, v_n) \geq 0$ and $\lambda \in \mathbb{R}$; and~~

(iv) $v^T u > 0$.

Then $\lambda \leq \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(C')_{ij} = (C)_{ij} \qquad \text{for } 1 \leq i, j \leq n \ \underbrace{\text{~~with } v_i \neq 0 \text{ and } u_j \neq 0\text{~~}}_{\text{if assume } C \text{ irreducible}}.$$

□

Remark

The restricted form is the special case of the general form by applying

$$P = I, \qquad Q = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \ddots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

Many other cases should be continuously investigated.

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Thank you for your attention.