Sharp bounds of the A_α-spectral radius of mixed trees

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Contents

Introductions

Mixed graphs

- A **mixed graph** is a graph in which both directed arcs and undirected edges between two distinct vertices may exist **at most once**.
- The **size** of a mixed graph is defined to be the number of directed arcs plus twice the number of undirected edges.

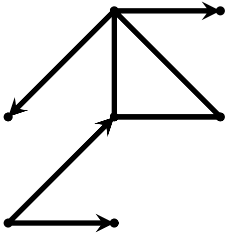


Figure: A mixed graph with size 10

A_α matrix and A_α-spectral radius of a mixed graph G

- Let $V(G) = \{1, 2, \ldots, n\}$ and $E(G)$ collects all directed arcs and undirected edges of the mixed graph G.
- $A(G) = (a_{ij})$ is the adjacency matrix of G where $a_{ij} = 1$ if and only if ij is an undirected edge or $\overrightarrow{ij}$ is a directed arc in $E(G)$.
- $D^+(G) = \mathrm{diag}(d_1^+, d_2^+, \ldots, d_n^+)$ is the out-degree matrix of G where $d_i^+ = |\{j : \overrightarrow{ij} \in E(G) \text{ or } ij \in E(G)\}|$.
- For $\alpha \in [0, 1]$, define

$$A_\alpha(G) = \alpha D^+(G) + (1 - \alpha)A(G).$$

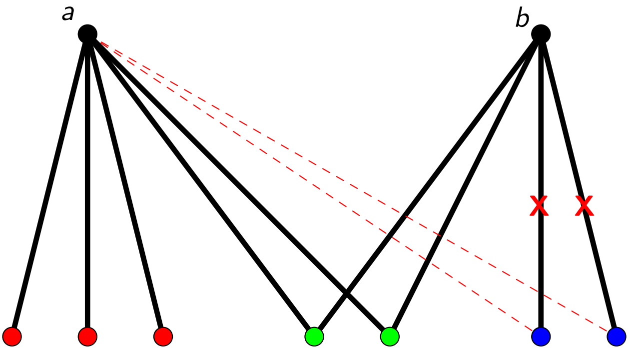
- The **A_α-spectral radius** $\rho_\alpha(G)$ of G is the spectral radius of $A_\alpha(G)$.

History

- **A_α-matrices of undirected graphs** : [Nikiforov, 2017].
- **A_α-spectral radii of trees and unicyclic graphs** : [Li, Chen, Meng, 2019].
- **A_α-matrices of digraphs** : [Liu, Wu, Chen, Liu, 2019].

Kelmans transformation

Kelmans transformation (undirected graph)



$G \rightarrow G_b^a$

A.K. Kelmans, On graphs with randomly deleted edges, *Acta Math. Hungar* 37 (1981) 77–88.

A result of P. Csikvári

The largest real eigenvalues of adjacency matrices will not be decreased after a Kelmans transformation of an **undirected graph** G, i.e.

$$\rho(G_b^a) \geq \rho(G)$$

P. Csikvári, On a conjecture of V. Nikiforov, *Discrete Math.* 309 (2009) 4522–4526.

Kelmans transformation (matrix realization)

$$A(G) = (c_{ij}) \quad (c_{ij} \in \{0,1\})$$

$$\rightarrow \quad A(G_b^a) = \begin{array}{c} \\ i \\ \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} \quad j \quad & \quad a \quad & \quad b \quad \end{array} \\ \left[\begin{array}{ccc} & & \\ c_{ij} & c_{ia} + t_i & c_{ib} - t_i \\ & & \\ c_{aj} + s_j & 0 & c_{ab} \\ c_{bj} - s_j & c_{ba} & 0 \end{array} \right] \end{array}$$

For each $i, j \in [n] - \{a, b\}$, the above $t_i, s_j \in \{0,1\}$ satisfy

- $\max(0, c_{ib} - c_{ia}) \le t_i \le c_{ib}$,
- $\max(0, c_{bj} - c_{aj}) \le s_j \le c_{bj}$.

Generalizing to nonnegative matrices

Let $C = (c_{ij})$ be a nonnegative matrix of order n, **not necessary symmetric**, with $c_{ab} = c_{ba}$ for some $1 \leq a \neq b \leq n$. The **Kelmans transformation of C from b to a** (denoted as $C_b^a = C_b^a(t_i; s_i; k)$) is the following matrix

$$C_b^a = \begin{array}{c} \\ i \\ \\ a \\ b \end{array} \begin{array}{c} \begin{array}{ccc} j & a & b \end{array} \\ \left[\begin{array}{ccc} & & \\ c_{ij} & c_{ia} + t_i & c_{ib} - t_i \\ & & \\ c_{aj} + s_j & c_{aa} + k & c_{ab} \\ c_{bj} - s_j & c_{ba} & c_{bb} - k \end{array} \right] \end{array}$$

where $\max(0, c_{bb} - c_{aa}) \leq k \leq c_{bb}$ and for each $i, j \in [n] - \{a, b\}$:

- $\max(0, c_{ib} - c_{ia}) \leq t_i \leq c_{ib}$,
- $\max(0, c_{bj} - c_{aj}) \leq s_j \leq c_{bj}$.

Theorem on spectral radius

Let $C = (c_{ij})$ denote a nonnegative square matrix of order n such that $c_{ab} = c_{ba}$ for some $1 \leq a \neq b \leq n$. Choose k, t_i, s_i for $i \in [n] - \{a, b\}$ that satisfy $\max(0, c_{bb} - c_{aa}) \leq k \leq c_{bb}$ and for each $i, j \in [n] - \{a, b\}$:

- $\max(0, c_{ib} - c_{ia}) \leq t_i \leq c_{ib}$,
- $\max(0, c_{bj} - c_{aj}) \leq s_j \leq c_{bj}$.

Let $C_b^a = C_b^a(t_i; s_i; k)$ be the Kelmans transformation from b to a with respect to $(t_i; s_j; k)$. Then

$$\rho(C) \leq \rho(C_b^a).$$

□

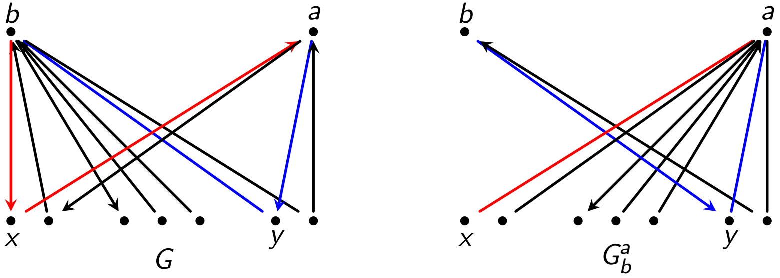
Ideal of the proof

Let $w = (w_i) > 0$ be left Perron vector for $\rho(C)$ of C. We prove that

$$\begin{aligned} w_a \geq w_b \quad &\Rightarrow \quad \rho(C_b^a) \geq \rho(C) \\ \rho(C_b^a) \quad &= \quad \rho(C_a^b) \end{aligned}$$

Kelmans transformation on mixed graphs

Let a, b be two vertices in mixed graph G such that a and b have either an undirected edge or have no arc. The Kelmans transformation G_b^a of G from b **to** a generalizes that of undirected graphs as illustrated below.



Kelman's transformation (A_α matrix)

Let $A = (c_{ij})$ be the adjacency matrix of a mixed graph G with $c_{ab} = c_{ba}$ for some $1 \leq a \neq b \leq n$. Let G^a_b be the Kelmans transformation of G from b to a. Then

$$A_\alpha(G^a_b) = A_\alpha(G)^a_b.$$

Corollary

Let G be a mixed graph with two specified vertices a, b which have no arc, and let $\alpha \in [0,1]$. Then the A_α-spectral radii $\rho_\alpha(G)$ and $\rho_\alpha(G_b^a)$ satisfy

$$\rho_\alpha(G) \leq \rho_\alpha(G_b^a).$$

Poset of mixed graphs

Poset of mixed graphs

Let $[G]$ denote the set of mixed graphs that are isomorphic to G and

$$\mathcal{G}(n, m) := \{[G] : G \text{ is a mixed graph of order } n \text{ and size } m\}.$$

We define the order $[G] \leq [G_b^a]$ and extend the order to be a partially ordered set on $(\mathcal{G}(n, m), \leq)$.

Poset of mixed trees

Let $n, m \in \mathbb{N}$ with $n-1 \leq m \leq 2n-2$,

$$\mathcal{T}(n,m) := \{[T] \in \mathcal{G}(n,m) \colon T \text{ is a mixed tree}\}.$$

To let the set $\mathcal{T}(n,m)$ be closed under the Kelmans transformations from b to a, **we need to choose a and b** in the following seven situations:

$$a - b,\ \ a - x \to b,\ \ a - x \leftarrow b,\ \ a \leftarrow x - b,$$
$$a \to x - b,\ \ a \to x \leftarrow b,\ \ a \leftarrow x \to b$$

Maximum elements of $\mathcal{T}(n, m)$

Proposition

Let $[T] \in \mathcal{T}(n, m)$. Then $[T]$ is a maximal element in $\mathcal{T}(n, m)$ if and only if

(i) T is a **mixed star**, or

(ii) T is a mixed tree without undirected edges (i.e. $m = n - 1$) and whenever the subgraph $a \to x \leftarrow b$ or $a \leftarrow x \to b$ appears in T, one of a and b is a leaf.

□

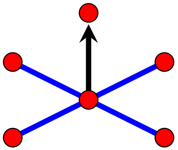
Main Theorem

If $\alpha \in [0,1]$ and T is a mixed tree of order n and size m, then

$$\rho_\alpha(T) \leq \frac{1}{2}\left(\alpha n + \sqrt{\alpha^2 n^2 - 4\alpha^2(n-1) + 4(1-\alpha)^2(m-n+1)}\right).$$

Moreover, every mixed star of order n and size m with maximum out-degree $n-1$ attains the upper bound.

$S_{6,9}$



$$n = 6,\ m = 9 \quad \Rightarrow \quad \rho_\alpha(T) \leq \rho_\alpha(S_{6,9})$$

Theorem

If T is a mixed tree of order n and size m, and set $k = \lceil \frac{n}{2n-m-1} \rceil$ then

$$\rho_\alpha(T) \geq \rho_\alpha(P_k).$$

Moreover, the lower bound is attained when $T = P_n$ ($m = 2n-2$, $k = n$). □

$$n = 6,\ m = 9, k = 3 \quad \Rightarrow \quad \rho_\alpha(T) \geq \rho_\alpha(P_3)$$

Thank you for your time and attention!