Classical Distance-Regular Graphs with $a_2 > a_1 = 0$

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Outline

1. Classical Distance-Regular Graphs
2. Classical Distance-Regular Graphs with $b < -1$
3. Distance-Regular Graphs that have d-Bounded Property
4. Two Special Cases $c_2 = 1$ and $a_2 > a_1 = 0$

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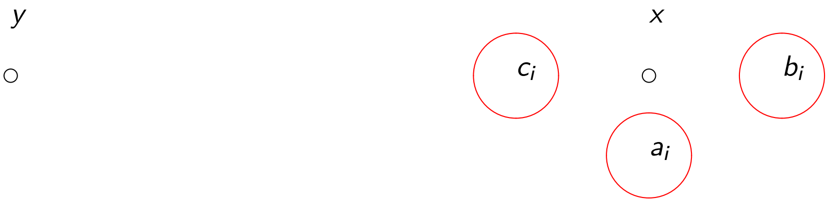
Classical Distance-Regular Graphs

Distance-regular graphs

A graph $G = (V, E)$ with diameter d is **distance-regular** if and only if for $i \leq d$, the numbers

$$\begin{aligned} c_i &:= |G_1(x) \cap G_{i-1}(y)|, \\ a_i &:= |G_1(x) \cap G_i(y)|, \\ b_i &:= |G_1(x) \cap G_{i+1}(y)| \end{aligned}$$

are constants subject to all vertices x, y with $\partial(x, y) = i$, called the **intersection numbers** of G.



Note that $a_i + b_i + c_i = b_0$ and $k := b_0$ is the **valency** of G.

Classical distance-regular graphs

A distance-regular graph G is said to have **classical parameters** (d, b, α, β) whenever d is the diameter of G and the intersection numbers of G satisfy

$$\begin{aligned} c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right) && \text{for } 0 \le i \le d, \\ b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right) && \text{for } 0 \le i \le d, \\ a_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(\beta - 1 + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b - \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right)\right) && \text{for } 0 \le i \le d. \end{aligned}$$

where b is an integer, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}_b := 0$ and

$$\begin{bmatrix} i \\ 1 \end{bmatrix}_b := 1 + b + b^2 + \cdots + b^{i-1} \quad \text{for } 1 \le i \le d.$$

It is known that $b \neq 0, -1$ (BCN book).

Remark

Classical distance-regular graphs (CDRGs) form a subclass of **Q-polynomial distance-regular graphs** (BI Book).

Many results concerning classical distance-regular graphs, which we will survey, may extend to a broader Q-polynomial case.

Theorem (The classification of CDRG with $b = 1$)

If G is a distance-regular graph with classical parameters (d, b, α, β), $d \geq 3$ and $b = 1$, then precisely one of the following (i)-(v) holds. (i) G is the Johnson graph $J(n, d)$. (ii) G is the Gosset graph $E_7(1)$ $(d = 3)$. (iii) G is the Hamming graph $H(d, n)$. (iv) G is the halved n-cube $\frac{1}{2}H(n, 2)$. (v) G is the Doob graph $(\alpha = 0, \beta = 3)$.

- Y. Egawa, Characterization of $H(n, q)$ by the parameters, *J. Combin. Theory Ser. A*, 31:108–125, 1981.
- A. Neumaier, Characterization of a class of distance-regular graphs, *J. Reine Angew. Math.*, 357:182–192, 1985.
- P. Terwilliger, Root systems and the Johnson and Hamming graphs, *European J. Combin.*, 8:73–102, 1987.
- P. Terwilliger. The classification of distance-regular graphs of type IIB, *Combinatorica*, 8:125–132, 1988.

Classical Distance-Regular Graphs with $b < -1$

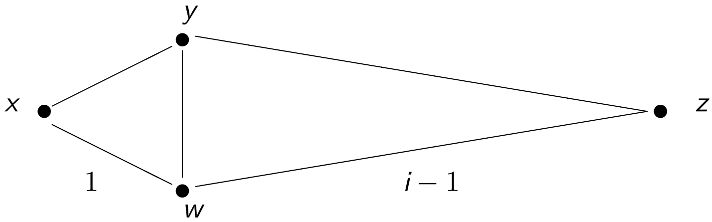
Theorem

Let G denote a distance-regular graph with classical parameters (d, b, α, β). Then the following (i)-(ii) hold.

(i) If G contains no kite of length 2, then G contains no kite of any length.

(ii) If $b < -1$ then G has no kite of any length.

- P. Terwilliger, Kite-free distance-regular graphs, *European J. Combin.*, 16 (1995), 405–414.

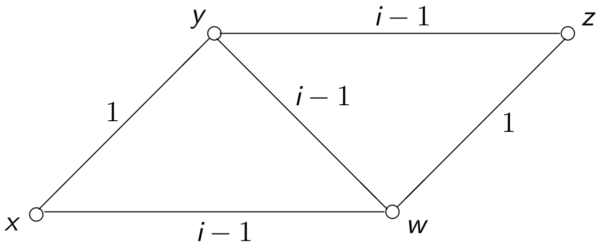


A kite of length *i*

Proposition

If a graph contains no parallelogram then it contains no kites. The converse is also true if the graph is a distance-regular graph with classical parameters and diameter $d \geq 3$.

- C. Weng, Weak-geodetically Closed Subgraphs in Distance-Regular Graphs, *Graphs and Combinatorics*, 14 (1998), 275–304.



A parallelogram of length i

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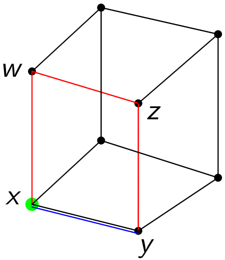
t-bounded distance-regular graphs

A distance-regular graph G is **t-bounded** if for any two vertices x, y at distance $\partial(x, y) \leq t$, there exists a distance-regular subgraph $\Delta = \Delta(x, y)$ of G containing x and y of diameter $\partial(x, y)$ such that

$$a_i(\Delta) = a_i(G), \quad c_i(\Delta) = c_i(G) \qquad (1 \leq i \leq \partial(x, y)).$$



$\Delta(x, x) \subseteq \Delta(x, y) \subseteq \Delta(x, z) \subseteq G.$

Theorem

If G is a *d-bounded* distance-regular graph with classical parameters (d, b, α, β), $d \geq 4$ and $b < -1$, then precisely one of the following (i)-(iii) holds:

(i) G **is the dual polar graph** ${}^2A_{2d-1}(-b)$.

(ii) G **is the Hermitian forms graph** $Her_{-b}(d)$.

(iii) $\alpha = \frac{b-1}{2}$, $\beta = \frac{-1-b^d}{2}$, **and** $-b$ **is a power of an odd prime.**

- C. Weng. Classical distance-regular graphs of negative type, *J. Combin. Theory Ser. B*, 76:93–116, 1999.

The next goal of this survey is about removing the above "***d*-bounded**" assumption.

Distance-Regular Graphs that have d-Bounded Property

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Theorem (Case $a_1 > 0$ and $c_2 > 1$)

Let G be a distance-regular graph with diameter d, intersection numbers $a_1 > 0$ and $c_2 > 1$. If G contains no parallelogram of any length, then G is d-bounded.

- C. Weng, Weak-geodetically Closed Subgraphs in Distance-Regular Graphs, *Graphs Combin.*, 14 (1998), 275–304.

Theorem (Case $a_2 > a_1 > 0$, $c_2 = 1$)

Let G be a distance-regular graph with diameter d, intersection numbers $a_2 > a_1 > 0$ and $c_2 = 1$. If G contains no parallelogram of any length, then G is d-bounded.

- H. Suzuki, On strongly closed subgraphs of highly regular graphs, *European J. Combin.*, 16 (1995), 197–220.

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Theorem (Case $a_2 > a_1 = 0$ and $c_2 > 1$)

Let G be a distance-regular graph with diameter $d \geq 3$, intersection numbers $a_2 > a_1 = 0$ and $c_2 > 1$. If G contains no parallelogram of any length, then G is d-bounded.

- A. Hiraki, Distance-Regular Graph with $c_2 > 1$ and $a_1 = 0 < a_2$, *Graphs Combin.*, 25 (2009), 65–79.

Theorem (Case $a_2 > a_1 = 0$ and $c_2 = 1$)

Let G be a distance-regular graph with diameter $d \geq 3$, intersection numbers $a_2 > a_1 = 0$ and $c_2 = 1$. If G contains no parallelogram of any length, then G is d-bounded.

- Y. Huang, Y. Pan, C. Weng, Nonexistence of a Class of Distance-regular Graphs, *Electron. J. Comb.*, 22 (2015), #P2.37.

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Lemma (Case $a_2 = a_1 = 0$, the bipartite graphs)

There is no classical distance-regular graph with $b < -1$ and $a_2 = a_1 = 0$.

Proof.

Note that

$$a_2 = (b+1)(a_1 - \alpha(b+1)) \quad \text{and} \quad c_2 = (1+\alpha)(1+b).$$

If $a_2 = a_1 = 0$, then $\alpha = 0$ and $b = c_2 - 1 \geq 0$, a contradiction. □

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Remark

If G is a Q-polynomial bipartite distance-regular graph with diameter $d \geq 14$, then precisely one of the following (i)-(iv) holds. (i) G is the ordinary $2d$-cycle. (ii) G is the Hamming cube $H(d,2)$. (iii) G is the antipodal quotient of $H(2d,2)$. (iv) G has classical parameters $(d,b,\alpha,\beta) = (d,b,0,1)$, which are realized by both bipartite dual polar graph $D_d(b)$ and its isomer Hemmeter graph.

- J. S. Caughman IV, Bipartite Q-polynomial distance-regular graphs, *Graphs Combin.*, 20 (2004), 47–57.

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Lemma

In a classical distance-regular graph,

$$a_2 = a_1 c_2 \quad \Rightarrow \quad a_i = a_1 c_i \quad (\textbf{for all } i).$$

Proof.

This follows by direct computation

$$a_i - a_1 c_i = (a_2 - a_1 c_2)\frac{\begin{bmatrix} i \\ 1 \end{bmatrix}_b \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}_b} \qquad (2 \leq i \leq d).$$

□

A distance-regular with no kite of length 2 and $a_i = a_1 c_i$ for $1 \leq i \leq d$ is called a **near $2d$-gon** (BCN, Theorem 6.4.1). A near $2d$-gon distance-regular graph is **thick** if $a_1 \neq 0$.

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Theorem

If G is a thick Q-polynomial near $2d$-gon distance-regular graph with diameter $d \geq 4$, then G is a Hamming graph $H(d, n)$ or a dual polar graph.

- B. Bruyn and F. Vanhove, On Q-polynomial regular $2d$-gons, *Combinatorica*, 35 (2015), 181–208.

Corollary (Case $a_2 = a_1 > 0$)

The is no classical distance-regular graph G with $d \geq 4$, $b < -1$ and $a_2 = a_1 > 0$.

- Y. Tian, C. Lin, B. Hou, L. Hou, S. Gao, Further study of distance-regular graphs with classical parameters with $b < -1$, *Discrete Math.*, 347 (2024), 113817.

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Summary of CDRGs with $b < -1$ and $d \geq 4$

- $a_2 = a_1$ (no such graphs)
  (B. Bruyn, et al., 2015; Y. Tian, et al., 2024);
- $a_2 > a_1$ (having d-bounded property):
  - $a_1 > 0$ and $c_2 = 1$ (H. Suzuki,1995);
  - $a_1 > 0$ and $c_2 > 1$ (C. Weng, 1998);
  - $a_1 = 0$ and $c_2 > 1$ (A. Hiraki, 2009);
  - $a_1 = 0$ and $c_2 = 1$ (Y. Hunag, et al., 2015).

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Corollary

If G is a distance-regular graph with classical parameters (d, b, α, β), $d \geq 4$ and $b < -1$, then precisely one of the following (i)-(iii) holds:

(i) **G is the dual polar graph ${}^2A_{2d-1}(-b)$.**

(ii) **G is the Hermitian forms graph $Her_{-b}(d)$.**

(iii) **$\alpha = \frac{b-1}{2}$, $\beta = \frac{-1-b^d}{2}$, and $-b$ is a power of an odd prime.**

Remark

- The Gewirtz graph has classical parameters $(d, b, \alpha, \beta) = (2, -3, -2, -5)$ that satisfies (iii) of the above corollary.
- The existence of a distance-regular graph other that the Gewirtz graph in the case (iii) of the above corollary remains open.

Two Special Cases

$c_2 = 1$ and $a_2 > a_1 = 0$

We will show that $b < -1$ in these two cases.

Theorem

Let G be a distance-regular graphs with classical parameters (d, b, α, β) and $d \geq 3$. Then $b \neq 0, -1$ is an integer, and the following (i)-(iii) are equivalent.

(i) $a_i = a_1 c_i \quad (0 \leq i \leq d)$.

(ii) $a_2 = a_1 c_2$.

(iii) $b \in \{c_2 - 1, -a_1 - 1\}$.

- A. Brouwer, A. Cohen, A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, 1989. (Proposition 6.2.1)
- C. Weng, Weak-geodetically Closed Subgraphs in Distance-Regular Graphs, *Graphs Combin.*, 14 (1998), 275–304. (Theorem 5.4)

In particular, there is no such G with $a_2 = a_1 = 0$ and $c_2 = 1$.

Theorem

If G is a distance-regular graph with classical parameters (d, b, α, β) with diameter $d \geq 3$ and intersection numbers $a_2 > a_1 = 0$, then $\alpha < 0$ and $b < -1$.

- Y. Pan, M. Lu and C. Weng, Triangle-free Distance-regular Graphs, *J. Algebraic Combin.*, 27 (2008), 23–34. (Lemma 3.3)

Lemma

Let G be a distance-regular graph with classical parameters (d, b, α, β) with $d \geq 3$, $\alpha \neq 0$, and $a_1 \neq 0$. If G has no kite of length 2, then $b < -1$.

- P. Terwilliger, Kite-free distance-regular graphs, *European J. Combin.*, 16 (1995), 405–414.
- C. Weng, Kite-Free P- and Q-Polynomial Schemes, *Graphs Combin.*, 11 (1995), 201-207.

Theorem

There exists no distance-regular graph G with classical parameters (d, b, α, β), intersection number $c_2 = 1$ and diameter $d \geq 4$.

Proof.

The assumption $1 = c_2 = (1+\alpha)(1+b)$ implies $\alpha \neq 0$ since $b \neq 0$. The case $a_2 = a_1 = 0$ and $c_2 = 1$ trivially does not exist. From the previous discussion, we have $b < -1$ in all possible cases, so we might apply the classification of graph with $b < -1$. The case $a_2 = a_1 > 0$ and $c_2 = 1$ does not exist follows from the classification of near $2d$-gon. The case $a_2 > a_1$ implies that the graph is d-bounded, and consequently $\alpha = \frac{b-1}{2}$, so $1 = c_2 = (1+b)^2/2$, a contradiction to b an integer. □

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Theorem

Let G be a distance-regular graph with diameter $d \geq 3$, intersection numbers $a_2 > a_1 = 0$. If G has classical parameters (d, b, α, β), then $c_2 \leq 2$.

- Y. Pan and C. Weng, Three bounded properties in triangle-free distance-regular graphs, *European J. Combin.*, 29 (2008), 1634–1642.
- Y. Pan and C. Weng, A note on triangle-free distance-regular graphs with a pentagon, *J. Combin. Theory Ser. B*, 99 (2009) 266–270.
- A. Hiraki, Distance-Regular Graph with $c_2 > 1$ and $a_1 = 0 < a_2$, *Graphs Combin.*, 25 (2009), 65–79.

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Corollary

Let G be a distance-regular graph with classical parameters (d, b, α, β), $d \geq 4$, and intersection numbers $a_2 > a_1 = 0$. Then either $(d, b, \alpha, \beta) = (d, -2, -3, -1 - (-2)^d)$ (the Hermitian forms graph $Her_2(d)$) or $(d, -3, -2, (-1 - (-3)^d)/2)$.

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Corollary

Let G be a distance-regular graph with classical parameters (d, b, α, β), $d \geq 4$, and intersection numbers $a_2 > a_1 = 0$. Then either $(d, b, \alpha, \beta) = (d, -2, -3, -1-(-2)^d)$ (the Hermitian forms graph $Her_2(d)$) or $(d, -3, -2, (-1-(-3)^d)/2)$.

Proof.

We have known $\alpha < 0$ and $b < -1$. Since $a_2 > a_1 c_2$, G is not the dual polar graph ${}^2A_{2d-1}(-b)$. Since the Hermitian forms graph $Her_{-b}(d)$ has $\alpha = b - 1$ and $\beta = -1 - b^d$, the assumption $0 = a_1 = \beta - 1 + \alpha\left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - 1\right)$ implies $(d, b, \alpha, \beta) = (d, -2, -3, -1-(-2)^d)$. The remaining case is $\alpha = (b-1)/2$ and $\beta = (-1-b^d)/2$. By the previous theorem, $c_2 \leq 2$. Since we have excluded the situation $c_2 = 1$, we have $(1+b)^2/2 = (1+\alpha)(1+b) = c_2 = 2$. Hence $(d, b, \alpha, \beta) = (d, -3, -2, (-1-(-3)^d)/2)$. □

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Remark

- The above corollary were proved by A. Hiraki in 2009 under an additional assumption $c_2 > 1$, but loosen the assumption $d \geq 4$ to $d \geq 3$.
- The existence of a distance-regular graph with classical parameters

$$(d, b, \alpha, \beta) = \left(d, -3, -2, \frac{-1-(-3)^d}{2}\right)$$

and $d \geq 3$ remains open.

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