The largest real eigenvalues of nonnegative matrices by applying Kelmans transformations

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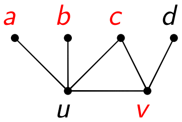
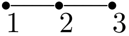
Undirected graphs

G:



$V(G) = \{u, v, a, b, c, d\}$
$E(G) = \{ua, ub, uc, uv, vc, vd\}$
$N(u) = \{a, b, c, v\}$
$N(v) = \{u, c, d\}$
$N[v] = \{u, c, d, v\}$

Adjacency matrices

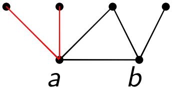
$G:$ 

$$\longrightarrow \qquad A(G) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Kelmans transformations for undirected graphs

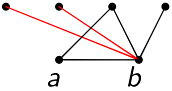
The graph $G(a, b)$ is obtained from G with the same vertex set and replacing edge ac by bc for each $c \in N(a) - N[b]$. □

G :



$\longrightarrow$

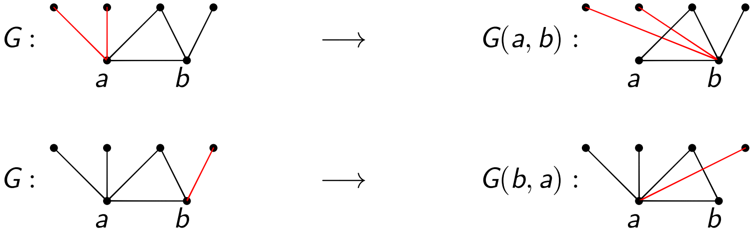
$G(a, b)$:



$$\begin{array}{c} \\ \\ \\ \\ b \\ a \end{array} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad (\text{unchanged})$$

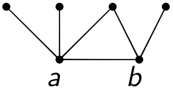
A.K. Kelmans, On graphs with randomly deleted edges, *Acta Math. Acad. Sci. Hung.* 37 (1981) 77-88.

Remark 1: $G(a, b)$ is isomorphic to $G(b, a)$



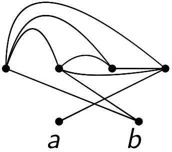
Complement graph

G :



$\longrightarrow$

$\overline{G}$:



$$\begin{matrix} \\ \\ \\ \\ \text{b} \\ \text{a} \end{matrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Remark 2: $\overline{G}(b,a)=\overline{G(a,b)}$

$$\begin{matrix} \\ \\ \\ \\ b \\ a \end{matrix}\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&1&1\\ 0&0&0&0&1&0\\ 0&0&1&1&0&1\\ 1&1&1&0&1&0 \end{pmatrix} \longrightarrow \quad G(a,b)=\begin{pmatrix} 0&0&0&0&1&0\\ 0&0&0&0&1&0\\ 0&0&0&0&1&1\\ 0&0&0&0&1&0\\ 1&1&1&1&0&1\\ 0&0&1&0&1&0 \end{pmatrix}$$

$$\begin{matrix} \\ \\ \\ \\ b \\ a \end{matrix}\begin{pmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&1&1\\ 0&0&0&0&1&0\\ 0&0&1&1&0&1\\ 1&1&1&0&1&0 \end{pmatrix} \longrightarrow \quad \overline{G}(b,a)=\begin{pmatrix} 0&1&1&1&0&1\\ 1&0&1&1&0&1\\ 1&1&0&1&0&0\\ 1&1&1&0&0&1\\ 0&0&0&0&0&0\\ 1&1&0&1&0&0 \end{pmatrix}$$

$$=\overline{G(a,b)}$$

Eigenvalues and eigenvectors

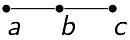
Let A be a real square matrix, u is a column vector, λ is a complex number. The number λ is an eigenvalue of A with associated eigenvector u if

$$Au = \lambda u.$$

A few facts from linear algebra

1. An $n \times n$ real symmetric matrix has n real eigenvalues and n corresponding orthonormal eigenvectors.
2. A nonnegative matrix M has a largest real eigenvalue $\rho(M)$, called the spectral radius of M.
3. The eigenvalue $\rho(M)$ is associated with a nonnegative eigenvector of length 1, called Perron vector.
4. All other eigenvalues λ of M satisfy $|\lambda| \leq \rho(M)$.
5. The spectral radius $\rho(G)$ of a graph G is the largest real eigenvalue of its adjacency matrix $A = A(G)$.

Example

$G:$ 

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ \pm 2 \\ \sqrt{2} \end{pmatrix} = \pm\sqrt{2} \begin{pmatrix} \sqrt{2} \\ \pm 2 \\ \sqrt{2} \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0.$$

Note that $\frac{1}{2\sqrt{2}}(\sqrt{2}, 2, \sqrt{2})$ is the Perron vector of A and the three eigenvectors

$$\frac{1}{2\sqrt{2}}(\sqrt{2}, 2, \sqrt{2}), \frac{1}{2\sqrt{2}}(\sqrt{2}, -2, \sqrt{2}), \frac{1}{\sqrt{2}}(1, 0, -1)$$

are orthonormal and $\rho(G) = \sqrt{2}$.

Rayleigh quotient for symmetric matrices

Lemma

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix. Then

$$\rho(A) = \max_{x \neq 0} \frac{x^t A x}{x^t x}.$$

Proof.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be eigenvalues of A with corresponding orthonormal eigenvectors $x_1, x_2, \ldots, x_n$. Write $x = \sum_{i=1}^n c_i x_i$, where $\sum_{i=1}^n c_i^2 = 1$. Then the maximum of

$$x^t A x = \sum_{i=1}^n c_i^2 \lambda_i$$

is λ_1 and is when $x = \pm x_1$. □

Non-symmetric matrices

$$\max_{x:x^tx=1} x^t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x = \max_{\substack{x=(a,b) \\ a^2+b^2=1}} ab = \frac{1}{2} \neq 0 = \rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Lemma

Let $x = (x_a)$ be the Perron vector of $A(G)$. Then

$$x_b \geq x_a \quad \Rightarrow \quad \rho(G) \leq \rho(G(a,b)).$$

Proof.

$$\begin{aligned}
\rho(G(a,b)) &= \max_{\|y\|_2=1} y^t A(G(a,b)) y \\
&\geq x^t A(G(a,b)) x = \sum_{cd \in E(G(a,b))} x_c x_d \\
&= \sum_{cd \in E(G)} x_c x_d + \sum_{e \in N(a)-N[b]} x_b x_e - x_a x_e \\
&\geq x^t A x = \rho(G).
\end{aligned}$$

□

An old result for undirected graphs

Theorem

$$\rho(G) \leq \rho(G(a,b))$$

Proof.

Let $x = (x_a)$ be the Perron vector of $A(G)$. Then

$$\begin{aligned} x_b \geq x_a \quad &\Rightarrow \quad \rho(G) \leq \rho(G(a,b)), \\ x_a \geq x_b \quad &\Rightarrow \quad \rho(G) \leq \rho(G(b,a)), \\ G(a,b)) \cong G(b,a) \quad &\Rightarrow \quad \rho(G(a,b)) = \rho(G(b,a)). \end{aligned}$$

□

Key fact in the above proof

$$G(a, b) \cong G(b, a)$$

Corollary

$$\rho(\overline{G}) \leq \rho(\overline{G(a,b)})$$

Proof.

$$\rho(\overline{G}) \leq \rho(\overline{G}(b,a)) = \rho(\overline{G(a,b)})$$

□

The above results $\rho(G) \leq \rho(G(a,b))$ and $\rho(\overline{G}) \leq \rho(\overline{G(a,b)})$ are from [Péer Csikvári, On a conjecture of V. Nikiforov, *Discrete Mathematics*, 309(13), 2009, 4522-4526]

A counterexample for digraph transformation

$$A = \begin{matrix} \\ \\ \mathsf{a} \\ \mathsf{b} \end{matrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & {\color{red}1} \\ 1 & 1 & {\color{blue}0} & {\color{blue}0} \\ 1 & 1 & {\color{blue}1} & {\color{blue}0} \end{pmatrix} \quad \rightarrow \quad A' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & {\color{red}1} & 0 \\ 1 & 1 & {\color{blue}0} & {\color{blue}0} \\ 1 & 1 & {\color{blue}1} & {\color{blue}0} \end{pmatrix} \ {\color{blue}\text{unchanged}}$$

$$\rho(A) \approx 2.234 > 2.148 \approx \rho(A')$$

We will define Kelmans transformations for nonnegative matrices.

Let C denote a nonnegative square matrix of order n. For a subset S of $[n] := \{1, 2, \ldots, n\}$ let $C[S]$ denote the principal submatrix of C restricted to S.

Fix $1 \leq a, b \leq n$.

Kelmans transformation of C from b to a (w.r.t. t_i, s_j)

$$C = \left(\begin{array}{ccc|cc} c & d & e & c_{14} & c_{15} \\ f & g & h & c_{24} & c_{25} \\ i & j & k & c_{34} & c_{35} \\ \hline c_{41} & c_{42} & c_{43} & \ell & m \\ c_{51} & c_{52} & c_{53} & n & o \end{array}\right) \begin{array}{c} \\ \\ \\ \text{a} \\ \text{b} \end{array} \geq 0$$

$$\to \quad C' = \left(\begin{array}{ccc|cc} c & d & e & c_{14}+t_1 & c_{15}-t_1 \\ f & g & h & c_{24}+t_2 & c_{25}-t_2 \\ i & j & k & c_{34}+t_3 & c_{35}-t_3 \\ \hline c_{41}+s_1 & c_{42}+s_2 & c_{43}+s_3 & \ell & m \\ c_{51}-s_1 & c_{52}-s_2 & c_{53}-s_3 & n & o \end{array}\right) \geq 0,$$

where $c_{ia} + t_i \geq c_{ib} - t_i$ and $c_{aj} + s_j \geq c_{bj} - s_j$.

Example

For $a = 4, b = 5, t_1 = 1, t_2 = t_3 = 0, s_1 = s_3 = 0, s_2 = 1$,

$$C = \begin{pmatrix} c & d & e & 0 & 1 \\ f & g & h & 1 & 0 \\ i & j & k & 1 & 1 \\ 1 & 0 & 1 & \ell & m \\ 0 & 1 & 1 & n & o \end{pmatrix} \quad \to \quad C' = \begin{pmatrix} c & d & e & 1 & 0 \\ f & g & h & 1 & 0 \\ i & j & k & 1 & 1 \\ 1 & 1 & 1 & \ell & m \\ 0 & 0 & 1 & n & o \end{pmatrix}.$$

For $a = 5, b = 4, t_2 = 1, t_1 = t_3 = 0, s_2 = s_3 = 0, s_1 = 1$,

$$C = \begin{pmatrix} c & d & e & 0 & 1 \\ f & g & h & 1 & 0 \\ i & j & k & 1 & 1 \\ 1 & 0 & 1 & \ell & m \\ 0 & 1 & 1 & n & o \end{pmatrix} \quad \to \quad C' = \begin{pmatrix} c & d & e & 0 & 1 \\ f & g & h & 0 & 1 \\ i & j & k & 1 & 1 \\ 0 & 0 & 1 & \ell & m \\ 1 & 1 & 1 & n & o \end{pmatrix} (= C'' \text{ later}).$$

C' and C''

Throughout let C' denote the Kelmans transformation of C from b to a with respect to t_i and s_j, and let $C'' = (c''_{ij})$ is the Kelmans transformation of C from a to b with respect to $c_{ia} - c_{ib} + t_i$ and $c_{aj} - c_{bj} + s_j$.

Symmetric subgraph

A matrix C is symmetric on $\{a, b\}$ of if the principal submatrix $C[a, b]$ is symmetric with constant diagonals. □

Example

The matrix

$$C = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & s & t \\ 0 & 1 & 1 & t & s \end{pmatrix}$$

is symmetric on $\{4, 5\}$.

Remark: C' is similar to C''

If C is symmetric on $\{a, b\}$, then C' is similar to C''.

$$C' = \left(\begin{array}{ccc|cc} c & d & e & c_{14}+t_1 & c_{15}-t_1 \\ f & g & h & c_{24}+t_2 & c_{25}-t_2 \\ i & j & k & c_{34}+t_3 & c_{35}-t_3 \\ \hline c_{41}+s_1 & c_{42}+s_2 & c_{43}+s_3 & s & t \\ c_{51}-s_1 & c_{52}-s_2 & c_{53}-s_3 & t & s \end{array}\right)$$

$$C'' = \left(\begin{array}{ccc|cc} c & d & e & c_{15}-t_1 & c_{14}+t_1 \\ f & g & h & c_{25}-t_2 & c_{24}+t_2 \\ i & j & k & c_{35}-t_3 & c_{34}+t_3 \\ \hline c_{51}-s_1 & c_{52}-s_2 & c_{53}-s_3 & s & t \\ c_{41}+s_1 & c_{42}+s_2 & c_{43}+s_3 & t & s \end{array}\right)$$

Main result

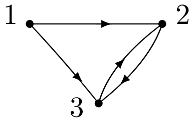
Theorem

If a nonnegative matrix C is symmetric on $\{a, b\}$, then $\rho(C) \leq \rho(C')$.

For the remaining of the talk, we will prove the above theorem.

Strongly connected

A digraph is strongly connected if for any two vertices x, y there exists a directed path from x to y. □



$$V(G) = \{1, 2, 3\}$$
$$E(G) = \{13, 12, 23, 32\}$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The above digraph is not strongly connected because there is no directed path from 2 to 1.

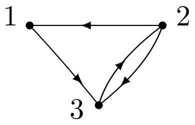
Digraph associated with a matrix

For an $n \times n$ matrix M, we define a digraph $G(M)$ with vertex set $\{1, 2, \ldots, n\}$ and edge set

$$E = \{xy \mid M_{xy} \neq 0\}.$$

$G(M)$ is called the digraph associated with M. □

$$M = \begin{pmatrix} 0 & 0 & -2 \\ 3 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \longrightarrow$$



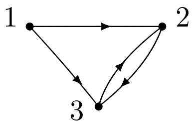
Irreducible matrix

A matrix is irreducible if its associated digraph is strongly connected. □

The matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is reducible since its associated digraph $G(A)$ is not strongly connected.



$$V(G(A)) = \{1, 2, 3\}$$
$$E(G(A)) = \{13, 12, 23, 32\}$$

A fact from linear algebra

A nonnegative irreducible matrix has a positive Perron vector.

The irreducible matrix A^ϵ

For a binary matrix $A = (a_{ij})$ and $0 < \epsilon < 1$, define the matrix $A^\epsilon = A + \epsilon J$, where J is the matrix with all 1's.

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad A^\epsilon = \begin{pmatrix} \epsilon & 1+\epsilon & 1+\epsilon \\ \epsilon & \epsilon & 1+\epsilon \\ \epsilon & 1+\epsilon & \epsilon \end{pmatrix}$$

The idea

Our proof is motivated by the following exercise.

Exercise

Suppose $0 \leq C \leq C'$. Show $\rho(C) \leq \rho(C')$.

Proof.

Let $u^t \geq 0$ be left Perron vector of C for $\rho(C)$ and $v^\epsilon > 0$ be right Perron vector of C'^ϵ for $\rho(C'^\epsilon)$. Then

$$\rho(C)u^t v^\epsilon = u^t C v^\epsilon \leq u^t C'^\epsilon v^\epsilon = \rho(C'^\epsilon)u^t v^\epsilon,$$

which implies $\rho(C) \leq \rho(C'^\epsilon)$ since $u^t v^\epsilon > 0$. Then by continuity

$$\rho(C) \leq \lim_{\epsilon \to 0^+} \rho(C'^\epsilon) = \rho(C').$$

□

The left Perron vector w^t for $\rho(C)$

Let $w^t = (w_i)$ denote the left Perron vector for $\rho(C)$ of C.
We will show that

$$\begin{aligned} w_a \geq w_b \quad &\Rightarrow \quad \rho(C) \leq \rho(C'), \\ w_b \geq w_a \quad &\Rightarrow \quad \rho(C) \leq \rho(C'') = \rho(C'). \end{aligned}$$

By symmetric, we might assume $w_a \geq w_b$ and only prove $\rho(C) \leq \rho(C')$.

The matrix $Q = I_n - E^{ba}$

Set $v^t = w^tQ$, where $Q = I_n - E^{ba}$ and E^{ba} denote the binary matrix of order n with a unique 1 in the position ba.

$$
\begin{aligned}
v^t =& (w_1, w_2, w_3, w_4 - w_5, w_5) \\
=& (w_1, w_2, w_3, w_4, w_5) \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ a \\ b \end{matrix} \\
=& w^tQ
\end{aligned}
$$

By the assumption $w_a \geq w_b$, $v^t \geq 0$

$v^t Q^{-1} C = w^t C = \rho(C) w^t = \rho(C) v^t Q^{-1}$

$$\begin{aligned} & v^t = w^t Q \quad \text{and} \quad w^t C = \rho(C) w^t \\ \Rightarrow \quad & v^t Q^{-1} C = \rho(C) v^t Q^{-1} \end{aligned}$$

$$Q^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} \\ \\ \\ a \\ b \end{matrix}$$

The matrix $Q^{-1}C(Q^{-1})^t$

$$
Q^{-1}C(Q^{-1})^t
= \left(\begin{array}{ccc|c}
c_{11} & \cdots & c_{1\ n-1} & c_{1n-1}+c_{1n} \\
\vdots & & \vdots & \vdots \\
c_{n-11} & \cdots & c_{n-1n-1} & c_{n-1n-1}+c_{n-1n} \\
\hline
c_{n-1\ 1}+c_{n1} & \cdots & c_{n-1\ n-1}+c_{nn-1} & c_{n-1n-1}+c_{nn-1} \\
 & & & +c_{n-1\ n}+c_{nn}
\end{array}\right)
\begin{array}{c} a \\ b \end{array}
$$

$Q^{-1}C(Q^{-1})^t \leq Q^{-1}C'^{\epsilon}(Q^{-1})^t$

$$C = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & {\color{red}1} \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{matrix} \\ \\ \text{a} \\ \text{b} \end{matrix} \quad \rightarrow \quad C' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & {\color{red}1} & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \quad Q^{-1}C(Q^{-1})^t = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix} \leq \begin{pmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 2 \end{pmatrix} = Q^{-1}C'(Q^{-1})^t$$

$$< Q^{-1}C'^{\epsilon}(Q^{-1})^t$$

The right eigenvector vector u^ϵ for $\rho(Q^t C'^\epsilon (Q^t)^{-1})$

Let $u^\epsilon \not< 0$ denote the right eigenvector vector for $\rho(Q^t C'^\epsilon (Q^t)^{-1})$.

(C'^ϵ is positive, but $Q^t C'^\epsilon (Q^t)^{-1}$ could have some negative entries)

Then

(i) $\rho(C'^\epsilon) = \rho(Q^t C'^\epsilon (Q^t)^{-1})$;

(ii) $C'^\epsilon (Q^{-1})^t u^\epsilon = C'^\epsilon (Q^t)^{-1} u^\epsilon = \rho(C'^\epsilon)(Q^t)^{-1} u^\epsilon = \rho(C'^\epsilon)(Q^{-1})^t u^\epsilon$;

(iii) $(Q^{-1})^t u^\epsilon > 0$ is an eigenvector for $\rho(C'^\epsilon)$.

Right multiplication of u^ϵ

$$Q^{-1}C(Q^{-1})^t u^\epsilon \leq Q^{-1}C'^\epsilon(Q^{-1})^t u^\epsilon = \rho(C'^\epsilon)Q^{-1}(Q^{-1})^t u^\epsilon.$$

Left multiplication of v^t

$$\rho(C)v^tQ^{-1}(Q^{-1})^tu^\epsilon = v^tQ^{-1}C(Q^{-1})^tu^\epsilon \leq v^t\rho(C^\epsilon)Q^{-1}(Q^{-1})^tu^\epsilon.$$

Deleting the positive term $v^t Q^{-1}(Q^{-1})^t u^\epsilon$

$$\begin{aligned} & \rho(C) v^t Q^{-1}(Q^{-1})^t u^\epsilon \leq \rho(C'^\epsilon) v^t Q^{-1}(Q^{-1})^t u^\epsilon \\ \Rightarrow\quad & \rho(C) \leq \rho(C'). \end{aligned}$$

The proof is completed. □

Thank you for your attention.