An introduction to
Hamiltonian Graph Theory
(Exercise)

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Exercise 1

If n points are placed in a plane with pairwise distances at least 1, then there are at most $3n$ unordered pairs of points at distance exactly 1.

Proof.

Let G be a graph with the n points as vertices and two vertices are adjacent if they have distance 1. Since there are at most 6 points in a unit circle with pairwise distances at least 1, G has maximum degree at most 6 Thus

$$2|E(G)| = \sum_{x \in V(G)} \deg(x) \leq 6n,$$

so $|E(G)| \leq 3n$ as desired. □

Exercise 2

Let G be a graph with vertex set $V(G) = \{1, 2, \ldots, n\}$ and $E(G) = \{ij : 0 < |i - j| \leq 2 \pmod{n}\}$. Show that if $n \geq 5$, then the edges of G can be partitioned into two Hamiltonian cycles.

Proof.

If we use $(1, 2, \ldots, n, 1)$ as the first Hamiltonian cycle, then the remaining edges also form a Hamiltonian cycle $(1, 3, \ldots, n, 2, 4, \ldots, n-1, 1)$ when n is odd. However, if n is even, the remaining edges form two cycles so we need to make a switch. In this case, we replace the two edges $(1, 2), (n-1, n)$ by $(n-1, 1), (n, 2)$ in the first cycle. Now the remaining edges form a Hamiltonian cycle. □

Exercise 3

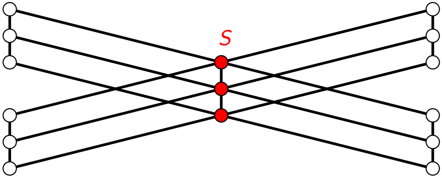
If T is a tree and G is a graph such that $G \Box T$ is Hamiltonian, then $|V(G)| \geq \Delta(T)$.

Proof.

If $\Delta(T) = \deg(s)$ and $G \Box T$ is Hamiltonian then

$$|V(G)| = |V(G \Box \{s\})| \geq c(G \Box T - G \Box \{s\}) = \Delta(T).$$

□



The graph $P_3 \Box K_{1,4}$ with $|V(P_3)| = 3 < 4 = |\Delta(K_{1,4})|$.

Exercise 4

Suppose $|E(G)| \geq \binom{n-1}{2} + 2$, where $n \geq 3$. Show that G is Hamiltonian.

Proof.

(Method 1) Suppose G is not Hamiltonian with degree sequence $d_1 \leq d_2 \leq \cdots \leq d_n$. Then $d_i \leq i$ and $d_{n-i} \leq n-i-1$ for some $i < n/2$ by Chvátal Theorem. Hence

$$
\begin{aligned}
|E(G)| =& \frac{1}{2}\left(\sum_{j=1}^{n} d_j\right) \\
\leq& \frac{1}{2}\left(\underbrace{i + \cdots + i}_{i} + \underbrace{(n-i-1) + \cdots + (n-i-1)}_{n-2i} + \underbrace{n-1 + \cdots + n-1}_{i}\right) \\
\leq& \binom{n-1}{2} + 1.
\end{aligned}
$$

□

Proof (Method 2)

For any two non-adjacent vertices u, v, the graph G' induced on $V(G) - \{u, v\}$ have size at most $\binom{n-1}{2}$. Hence

$$\deg(u) + \deg(v) \geq |E(G)| - \binom{n-2}{2} \geq \binom{n-1}{2} + 2 - \binom{n-2}{2} = n.$$

By Ore Theorem, G is Hamiltonian.

□

Proof (Method 3, without quoting theorems)

We prove by induction on n. If $n = 3$ then $|E(G)| = 3$ and $G = C_3$ is Hamiltonian.

Claim. If $G \neq K_n$ then there exists a vertex x with $n/2 \leq \deg(x) \leq n - 2$.
Proof of Claim. If there are $k \leq n - 1$ vertices of degree $n - 1$ and the remaining $n - k$ vertices have degree at most $n/2$, then $k \leq n/2$ and

$$[k(n-1) + (n-k)n/2]/2 \geq |E(G)| \geq \binom{n-1}{2} + 2,$$

a contradiction.

If $G = K_n$ then G is Hamiltonian. Suppose $G \neq K_n$. Use Claim to pick a vertex x with $n/2 \leq \deg(x) \leq n - 2$. Thus $|E(G - x)| \geq \binom{n-2}{2} + 2$. By induction $G - x$ has a Hamiltonian cycle C. Since $n/2 \leq \deg(x) \leq n - 2$ and C has $n - 1$ vertices, two adjacent vertices y, z in C are neighbors of x. Then $E(C) \cup \{xy, xz\} - \{yz\}$ is a Hamiltonian cycle of G.

□

Exercise 5

"If a simple graph G of order n contains two nonadjacent vertices whose degrees sum is at least n then G is Hamiltonian." Find a counterexample of the above statement.

Solution. A counter-example is G with $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{12, 23, 34, 41, 15\}$. Note that 1 and 3 are not adjacent with $\deg(1) + \deg(3) = 5 = |V(G)|$, but G is not hamiltonian since $\deg(5) = 1$.

Exercise 6

Find the first mistake in the following proof of the statement "If a simple graph G of order n contains two nonadjacent vertices whose degrees sum is at least n then G is Hamiltonian."

Wrong Proof.

(i) If the statement is false then there is a counter-example G with maximal number of edges.

(ii) There are two nonadjacent vertices u, v in G whose degrees sum is at least n.

(iii) Since $G + uv$ is not a counter-example, it is Hamiltonian,

(iv) so we have a Hamiltonian path $u = u_1, \ldots, u_n = v$ in G.

(v) Then uu_{i+1} and $u_i v$ are edges for some $1 \leq i \leq n-1$.

(vi) Hence $u_1, u_{i+1}, u_{i+2}, \cdots, u_n, u_i, u_{i-1}, \cdots, u_1$ is a Hamiltonian cycle.

□

Solution

(iii) is a mistake since the assumption that G is a counter-example with maximal number of edges does not imply $G + uv$ is Hamiltonian. For example, A maximal counter-example is G with $V(G) = \{1, 2, 3, 4, 5\}$ and $EG = \{12, 23, 34, 41, 24, 15\}$. G is a maximal counter-example since the only possible non adjacent vertices in $G + 13$ are $5, i$ with $i \neq 1$ and $\deg(5) + \deg(i) = 4 < 5 = |V(G + 13)|$. G satisfies the assumption since $u = 1$ and $v = 3$ are not adjacent in G with $\deg(u) + \deg(v) = 5 = |V(G)|$. $G + 13$ is not hamiltonian since $\deg(5) = 1$ in $G + 13$.

Exercise 7

A connected and locally $(k-1)$-connected graph G is k-connected.

Proof.

Assume G is not k-connected. Pick a subset $S \subseteq VG$ of minimal size such that $G - S$ is disconnected. Then $|S| \leq k-1$. Pick $v \in S$. Since $G - (S - v)$ is connected, there exist $u, w \in G_1(v)$ in different components of $G - S$. Then u, w are in different components of $G_1(v) \cap S$, a contradiction. □

Exercise 8

On a chessboard, a knight can move from one square to another if they differ by 1 in one coordinate and 2 by another. A knight-tour is a path that a knight visiting every single square exactly once and return to the starting square. Show that a 4-by-n chessboard contains no knight-tour for all n.

Proof.

First we color the squares by black and white alternately. Note that a knight can move to a black square only from a white square and vice versa. Now if we delete all n black squares from the middle two lines, the n white squares on the first and fourth line becomes n isolated square. Therefore

$$|c(G - S)| \geq n + 1 > n = |S|,$$

which implies a spanning cycle does not exist. □

Exercise 9

Let H denote a 2-connected and non-hamiltonian graph. Then H contains a subdivision of $K_{3,2}$.

Proof.

Let C be a cycle of maximum length in H, and yx be an edge with $x \in C$ and $y \notin C$. Let xz and xu be edges in C. The 2-connected assumption of H implies a cycle C_1 containing the two edges yx and xz. If C and C_1 only intersect in the edge xz, by joining the two paths in C and C_1 without the common edge xz, we will have a cycle of length greater than the length of C, a contradiction. Assume besides the edge xz, C and C_1 intersect at a third vertex. Let v be the first vertex in C from y along a path P as a portion of C_1 not containing the edge xz. Note that $v \neq u$; otherwise we have a longer cycle by replacing the edge ux by the path P and edge yx. Then the edge xy, and the edges in P and C form a subdivision of $K_{3,2}$. □

Exercise 10

If H is a graph that does not contain a subdivision of $K_{3,2}$, then the following are equivalent.

(i) **H is hamiltonian.**

(ii) **H is 1-tough.**

(iii) **H is 2-connected.**

Proof.

The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are clear. To prove (iii)$\Rightarrow$(i), let H be 2-connected and do not contain a subdivision of $K_{3,2}$. Applying last Exercise, we known H is hamiltonian. □

Exercise 11

If G is a planar, then the following are equivalent.

(i) **G is locally hamiltonian.**

(ii) **G is locally 1-tough.**

(iii) **G is locally 2-connected.**

Proof.

Let $x \in VG$ and apply the subgraph $H = G_1(x)$ to the previous Exercise. Note that $G_1(x)$ does not contain a subdivision of $K_{3,2}$ since $\{x\} \cup G_1(x)$ does not contain a subdivision of $K_{3,3}$. □