The degree pairs of a graph (Preliminary Report)

Chih-wen Weng

Department of Applied Mathematics, National Chiao Tung University

10:30-11:30, July 15, 2015

Let G be a simple connected graph with vertex set $VG = \{1, 2, \dots, n\}$ and edge set EG. Let d_i and m_i be the degree and average 2-degree of the vertex $i \in VG$ respectively, define as follows.

$$d_i := |N(i)|,$$
$$m_i := \frac{1}{d_i} \sum_{ji \in EG} d_j,$$

where $N(i)$ means the set $\{j \in VG \mid ji \in EG\}$ of neighbors of i.

The sequence of degree pairs (d_i, m_i)

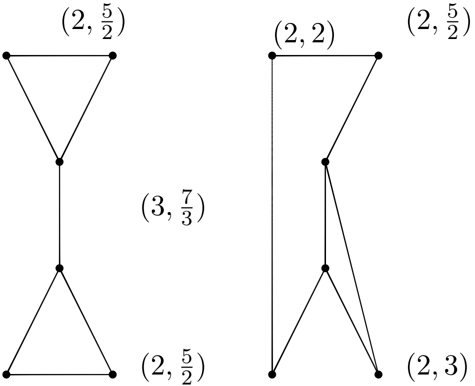


Figure: Two graphs whose sequences of degree pairs (d_i, m_i) are different.

The pair (d_i, m_i)

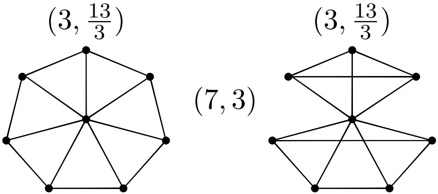


Figure: Two graphs have the same sequence of degree pairs.

Motivation

A graph G is k-regular if $d_i = k$ for all vertices $i \in VG$, and is pseudo k-regular if $m_i = k$ for all vertices $i \in VG$.

Motivation

In a two-side communication network, a node i of course knows the number d_i of nodes which are adjacent to i.

A node i might not know exactly how may nodes adjacent to each of its neighbors, but has rough idea of the mean number m_i of neighbors of its adjacent nodes.

Motivation

The pair (d_i, m_i) appears often in the study of maximum eigenvalue $\ell_1(G)$ of the Laplacian matrix $L = D - A$ associated with G.

(i) In 1998, Merris gave the following bound [1998M]:

$$\ell_1(G) \leq \max_{i \in VG} \{d_i + m_i\}.$$

(ii) Also in 1998, Li and Zhang gave the following bound [1998LZ]:

$$\ell_1(G) \leq \max_{ij \in EG} \left\{ \frac{d_i(d_i + m_i) + d_j(d_j + m_j)}{d_i + d_j} \right\}.$$

(iii) In 2001, Li and Pan gave the following bound [2001LP]:

$$\ell_1(G) \leq \max_{i \in VG} \left\{ \sqrt{2d_i(d_i + m_i)} \right\}.$$

(iv) In 2004, Das gave the following bound [2004D]:

$$\ell_1(G) \leq \max_{ij \in EG} \left\{ \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2} \right\}.$$

Motivation

(v) Also in 2004, Zhang gave the following bounds [2004Z]:

(va)

$$\ell_1(G) \le \max_{ij \in EG} \left\{ 2 + \sqrt{d_i(d_i + m_i - 4) + d_j(d_j + m_j - 4) + 4} \right\}.$$

(vb)

$$\ell_1(G) \le \max_{i \in VG} \left\{ d_i + \sqrt{d_i m_i} \right\}.$$

(vc)

$$\ell_1(G) \le \max_{ij \in EG} \left\{ \sqrt{d_i(d_i + m_i) + d_j(d_j + m_j)} \right\}.$$

Motivation

For this moment, we rearrange the vertices of G by $1, 2, \cdots, n$ such that $m_1 \geq m_2 \geq \cdots \geq m_n$. Let $a_1(G)$ is the maximum eigenvalue of adjacency matrix A associated with G. Then

(i) $a_1(G) \leq m_1$. (A simple application of Perron-Frobenius Theorem)

(ii) (2011, Chen, Pan and Zhang [2011CPZ]) Let $a := \max\{d_i/d_j \mid 1 \leq i, j \leq n\}$. Then

$$a_1(G) \leq \frac{m_2 - a + \sqrt{(m_2 + a)^2 + 4a(m_1 - m_2)}}{2}.$$

(iii) (2014, Huang and Weng [2014HW]) For any $b \geq \max\{d_i/d_j \mid ij \in EG\}$ and $1 \leq \ell \leq n$,

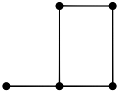
$$a_1(G) \leq \frac{m_\ell - b + \sqrt{(m_\ell + b)^2 + 4b\sum_{i=1}^{l-1}(m_i - m_\ell)}}{2}.$$

This talk emphasizes more on combinatorics than linear algebra.

It is easy for a graph (resp. a pair of prime numbers) to generate its sequence of degree pairs (resp. its product), but much harder for the reverse.

Can we determine which graphs G to have the prescribed sequence of the pairs $(d_i(G), m_i(G)) = (d_i, m_i)$.

$$\begin{pmatrix} d_i \\ m_i \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 2 & 1 \\ \frac{5}{3} & \frac{5}{2} & \frac{5}{2} & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 2 & 2 & 2 & 1 \\ 2 & \frac{5}{2} & \frac{5}{2} & 2 & 2 \end{pmatrix}$$



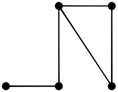


Figure: Two graphs uniquely determined by their sequences of degree pairs.

A feasible condition

Lemma 0.1

$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2$.

Proof.

$$\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i \frac{\sum_{ji \in EG} d_j}{d_i} = \sum_{j \in VG} \sum_{ij \in EG} d_j = \sum_{j \in VG} d_j^2.$$

□

Another feasible condition

Like a property of degree sequence, we have the following.

Lemma 0.2

There are even number of odd values $d_i m_i$ among $i \in VG$.

Proof.

Since $\sum_{i \in VG} d_i$ is even, there are even number of odd d_i, and so does d_i^2. Hence $\sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2$ is even. □

Corollary 0.3

$$\sum_{i \in VG} m_i^2 \geq \sum_{i \in VG} d_i^2$$

with equality iff $m_i = d_i = k$ for all i.

Proof.

$$\Big(\sum_{i \in VG} d_i^2\Big)\Big(\sum_{i \in VG} m_i^2\Big) \geq \Big(\sum_{i \in VG} d_i m_i\Big)^2 = \Big(\sum_{i \in VG} d_i^2\Big)^2$$

and equality iff $m_i = cd_i$, where $c = 1$ by the above lemma. This is also equivalent to that all neighbors of a vertex of minimum degree k also have degree k. □

Degrees give hints of graph properties, e.g. $\sum_{i \in VG} d_i = 2|EG|$.

Degree pairs give more of the graph structure.

Proposition 0.4

If $\max_{i \in VG} d_i m_i \geq n$ then the graph has girth at most 4.

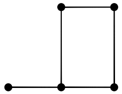
Proof.

If the graph has girth at least 5 then

$$n - 1 = |VG| - 1 \geq |G_1(i)| + |G_2(i)| = d_i m_i.$$

for any $i \in VG$. □

$$\begin{pmatrix} d_i \\ m_i \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 2 & 1 \\ \frac{5}{3} & \frac{5}{2} & \frac{5}{2} & 2 & 3 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 2 & 2 & 2 & 1 \\ 2 & \frac{5}{2} & \frac{5}{2} & 2 & 2 \end{pmatrix}$$



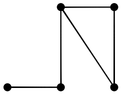


Figure: Two graphs uniquely determined by their sequence of degree pairs.

$$\max d_i m_i \geq 5 = n \qquad \Rightarrow \qquad \exists K_3 \text{ or } C_4.$$

Let G^2 be the square of G, i.e.

$$VG^2 = VG \text{ and } EG^2 = \{xy \mid d(x, y) \leq 2\}.$$

The coloring of G^2 applies to solve data aggregation problem and collision avoidance problem in a wireless sensor network G.

Using probability method, we have the following.

Proposition 0.5

$$\alpha(G^2) \geq \sum_{i \in VG} \frac{1}{1 + d_i m_i},$$

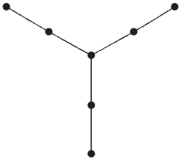
where $\alpha(G^2)$ *is the independence number of the square of G.*

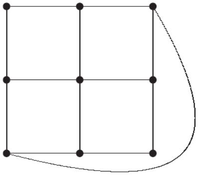
Proof.

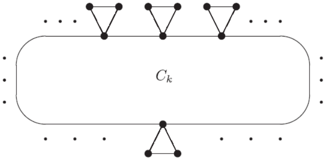
If a vertex is picked equally in random then the probability of a vertex i appears before those vertices in $G_1(i) \cap G_2(i)$ is $(1 + |G_i(i)| + |G_2(i)|)^{-1}$. Hence the expected size of a set consisting of these i is $\sum_{i \in VG}(1 + |G_i(i)| + |G_2(i)|)^{-1}$, which is at least $\sum_{i \in VG} \frac{1}{1+d_i m_i}$. □

We now turn to the study of pseudo k-regular graph, i.e. $m_i = k$ for all i.

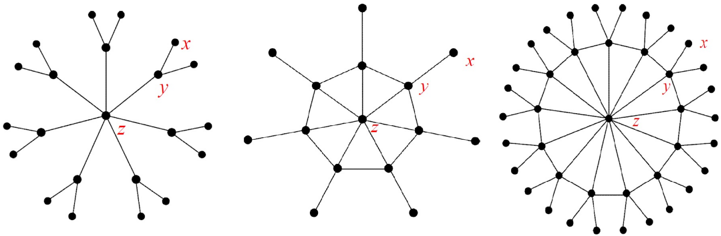
Pseudo 2-regular graph and pseudo 3-regular graphs







Pseudo k-regular graphs for $k = 3, 4, 5$



We try to find some theories for pseudo k-regular graphs.

From the definition of pseudo k-regular graphs, $k \in \mathbb{Q}$, but indeed we have the following.

Proposition 0.6

If G is pseudo k-regular then $k \in \mathbb{N}$.

Proof.

Let A be the adjacency matrix of G, and note that

$$(d_1, d_2, \ldots, d_n)A = k(d_1, d_2, \ldots, d_n).$$

Being a zero of the characteristic polynomial of A, k is an algebraic integer. Since k is also a positive rational number, k is indeed a positive integer. □

It is natural to ask when a pseudo k-regular graph attains the maximum number of edges when the order n of a graph is given.

Theorem 0.7

A pseudo k-regular graph has at most $nk/2$ edges, and the maximum is obtained iff the graph is regular.

Proof.

From

$$2k|EG| = \sum_{i \in VG} d_i m_i = \sum_{i \in VG} d_i^2 \geq (\sum_{i \in VG} d_i)^2/n = 4|EG|^2/n,$$

we have $|EG| \leq nk/2$ and equality is obtained iff d_i is a constant. □

The next is to ask when a pseudo k-regular graph attains the minimal number of edges when the order n of a graph is given.

Definition 0.8

Let T_k be the tree of order $k^3 - k^2 + k + 1$ whose root has degree $k^2 - k + 1$ and each neighbor of the root has $k - 1$ children as leafs.

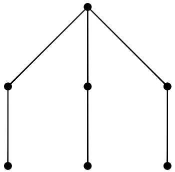


Figure: The tree T_2.

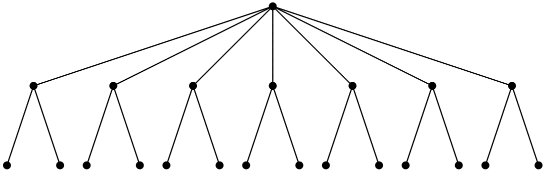


Figure: The tree T_3.

The first two cases of pseudo k-regular graphs are easy to settle.

Lemma 0.9

If G is connected pseudo 1-regular then G is K_2. □

Lemma 0.10

If G is connected pseudo 2-regular then G is a cycle or T_2.

Proof.

Note that $\Delta(G) = 2$ or 3, and the first implies that G is a cycle and the latter implies that $G = T_2$. □

We shall study the connected pseudo k-regular graphs of order n which attain the minimum number of edges, i.e. pseudo k-regular trees if it exists.

We also want to find a connected pseudo k-regular graph of order n whose maximum degree is maximal among all connected pseudo k-regular graph of order n.

It turns out that both problems have the same graph as their solutions.

The following is a technical but useful proposition.

Lemma 0.11

$$d_i \leq m_i(m_j - 1) + 1$$

for any j with $ji \in EG$ and $d_j \leq m_i$. Moreover the above equality holds iff $d_j = m_i$ and all neighbors of j have degree 1 except the neighbor i of j.

Proof.

Pick j such that $ji \in EG$ and $d_j \leq m_i$. Then $d_j m_j \geq d_i + (d_j - 1) \cdot 1$. Hence

$$m_i(m_j - 1) + 1 \geq d_j(m_j - 1) + 1 \geq d_i.$$

□

Theorem 0.12

Let G be a connected graph with $m_i \leq k$ (for example G is a pseudo k-regular graph) for all $i \in VG$, where $k \in \mathbb{N}$. Then

$$\Delta(G) \leq k^2 - k + 1.$$

Moreover the following (i)-(iv) are equivalent.

(i) $\Delta(G) = k^2 - k + 1.$

(ii) *G is the tree T_k.*

(iii) *G is a pseudo k-regular tree.*

(iv) *G has a vertex j such that $d_j = m_j = k$ and all neighbors of j have degree 1 with one exception.*

Proof of the Theorem 0.12

Choose i such that $d_i = \Delta(G)$. Then by Lemma 0.11, $\Delta(G) = d_i \leq m_i(m_j - 1) + 1 = k^2 - k + 1$ for any j with $ji \in EG$ and $d_j \leq m_i$. Moreover $\Delta(G) = k^2 - k + 1$ iff $d_j = m_j = m_i = k$ and $d_z = 1$ for all neighbors $z \neq i$ of j. Hence (i) and (ii) are equivalent.

The implications of (ii)⇒(iii) and (iii)⇒(iv) are clear.

Assume that (iv) holds, and let i be the unique neighbor of j with degree $d_i \neq 1$. Then $k^2 = d_j m_j = (k-1) + d_i$ to conclude that $d_i = k^2 - k + 1$. By the first statement of the theorem, $\Delta(G) = k^2 - k + 1$. This proves (i). □

Just ten days ago when we presented the last theorem in the conference "2015 International Conference on Graph Theory and Combinatorics & Eighth Cross-strait Conference on Graph Theory and Combinatorics", Professor Hou, Yaoping (Hunan Normal University) told us that the theorem had been proved in the following article:

S. Grünewald, Harmonic Trees, *Applied Mathematics Letters*, 15 (2002), 1001-1004.

Pseudo regular graphs are called harmonic graphs in that paper.

Let G be a pseudo k-regular graph.

The unique neighbor of a vertex of degree 1 of course has degree k in G.

We have seen in the previous proof that any neighbor of a vertex of degree $k^2 - k + 1$ also has degree k in G.

We are interested in what other vertices have their neighbors of the same degree k.

Lemma 0.13

Let G be a pseudo k-regular graph. Let ij be an edge with $2 \leq d_j < k$. Then

$$2 \leq d_i \leq k^2 - 3k + 4,$$

with the second equality iff all neighbors of j except i have degree $d_j = 2$.

Proof.

(i) is clear.
Note that $d_i \neq 1$, otherwise $d_j = k$, a contradiction. Indeed $d_z \neq 1$ for any neighbors z of j. Hence

$$d_i + 2(d_j - 1) \leq d_j m_j = d_j k.$$

Hence

$$d_i \leq d_j(k-2) + 2 \leq k^2 - 3k + 4.$$

□

Corollary 0.14

Let G be a pseudo k-regular graph of order n with a vertex of degree $d_i \geq k^2 - 3k + 5$. *Then*

(i) *Every neighbor j of i has degree* $d_j = k$;

(ii) *The order of G is at least* $f(k) := \lceil (5k^4 - 31k^3 + 94k^2 - 140k + 100)/k^2 \rceil$.

Note that for $k = 3$, $k^2 - 3k + 5 = 5$ and $f(3) = 11$.

Proof

(i) From Lemma 0.13(i) $d_j \neq 1$, and from Lemma 0.13(ii) $d_j \geq k$. This is true for all neighbors j of i. Hence $d_j = k$.

Proof

(ii) From $\sum_{w\in VG} d_w^2 = \sum_{w\in G} d_w m_w$,

$$d_i^2 + d_i k^2 + \sum_{w\notin\{i\}\cup G_1(i)} d_w^2 = kd_i + k^2 d_i + \sum_{w\notin\{i\}\cup G_1(i)} kd_w.$$

Hence

$$\begin{aligned} k^4 - 7k^3 + 22k^2 - 35k + 25 &\leq \sum_{w\notin\{i\}\cup G_1(i)} d_w(k - d_w) \\ &\leq \left(\frac{k}{2}\right)^2 (n - 1 - (k^2 - 3k + 5)). \end{aligned}$$

□

The family $\mathcal{E}_k$ of pseudo k-regular graphs

Let $\mathcal{E}_k$ be a family of graphs constructed as the following. Firstly pick a bipartite $(k-1)$-regular graph of order $2(2k-1)$ with bipartition $X \cup Y$, where $|X| = |Y| = 2k-1$. Then add a new vertex connecting to all vertices of X. One can check that graphs in $\mathcal{E}_k$ are pseudo k-regular of order $4k-1$ with maximum degree $2k-1$.

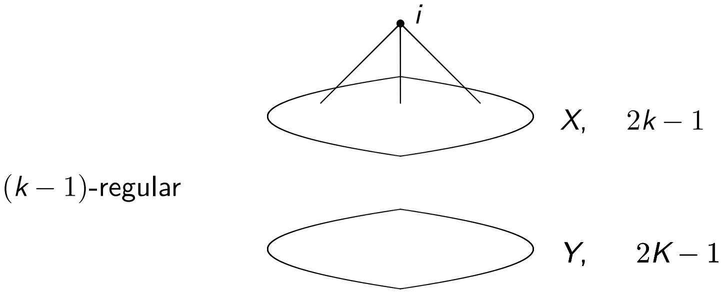
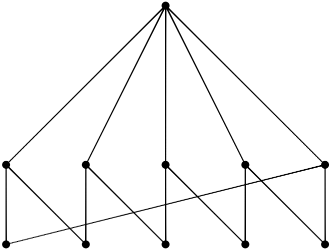


Figure: The graphs in $\mathcal{E}_k$.



From Corollary 0.14(ii), we know a pseudo 3-regular graph with maximum degree at least 5 has at least $f(3) = 11$ vertices. All the graphs in $\mathcal{E}_3$ are extremal for this property.

Pseudo 3-regular graphs

Now we restrict our attention to pseudo 3-regular graph G.

Note that the maximum degree $3 \leq \Delta(G) \leq k^2 - k + 1 = 7$ and the case $\Delta = 7$ is solved in by Theorem 0.12.

Lemma 0.15

Let G be a pseudo 3-regular graph with a vertex i of degree $d_i = 6$. Then all neighbors j of i have degree $d_j = 3$, and the neighbors of j have degree sequence $(6, 1, 2)$. □

Lemma 0.16

Let G be a pseudo 3-regular graph with a vertex i of degree $d_i = 5$. Then all neighbors j of i have degree $d_j = 3$, and the neighbors of j have degree sequence $(5, 2, 2)$ or $(5, 3, 1)$. □

Lemma 0.17

Let G be a pseudo 3-regular graph. Then the neighbor degree sequence of a vertex of degree 4 is $(3, 3, 3, 3)$, $(4, 3, 3, 2)$, or $(4, 4, 2, 2)$.

Pseudo 3-regular graph of order at most 10

We have shown that a pseudo 3-regular graph with maximum degree at least 5 must have at least 11 vertices.

We will list all the pseudo 3-regular graph of order at most 10.

Lemma 0.18

Let G be a connected pseudo 3-regular graph with $\Delta(G) = 4$ *and*

$$a_j := |\{i \in VG \mid d_i = j\}|$$

for $j = 1, 2, 3, 4$. *Then*

(i) $a_1 + a_2 = 2a_4$,

(ii) $|VG| = a_3 + 3a_4$,

(iii) $a_1 \leq a_3$,

(iv) a_1, a_2, a_3 *have same parity.*

Proof.

(i) and (ii) follow from solving

$$0 = \sum_{i \in VG} (m_i - d_i) d_i = \sum_{i \in VG} (3 - d_i) d_i = a_1 \cdot 2 + a_2 \cdot 2 + a_4(-4).$$

(iii) follows since there exists an injection from the set of degree one vertices into set of degree 3 vertices. Since there are even number of vertices of odd degrees, $a_1 + a_3$ is even. The remaining follows from (i) and (ii). This proves (iv). □

$|VG| = 7$:

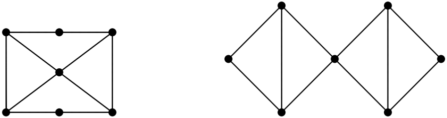


Figure: Graphs with sequence $(n, a_4, a_4, a_3, a_2) = (7, 1, 4, 2, 0)$.

$|VG| = 8$:

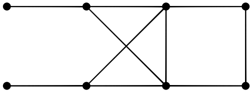


Figure: The graph with sequence $(n, a_4, a_4, a_3, a_2) = (8, 2, 2, 2, 2)$.

$|VG| = 9$(i):

Figure: Graphs with sequence $(n, a_4, a_4, a_3, a_2) = (9, 1, 6, 2, 0)$.

$|VG| = 9$(ii):

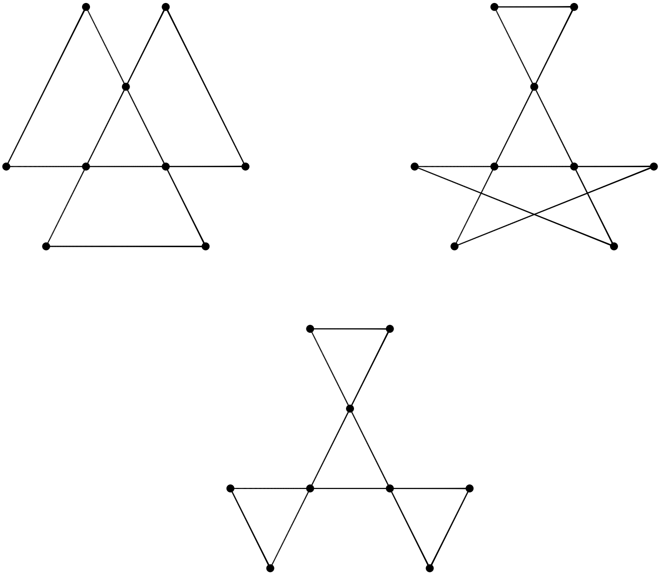


Figure: Graphs with sequence $(n, a_4, a_4, a_3, a_2) = (9, 3, 0, 6, 0)$.

$|VG| = 9$(iii):

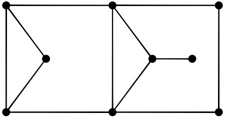


Figure: The graph with sequence $(n, a_4, a_4, a_3, a_2) = (9, 2, 3, 3, 1)$.

$|VG| = 10$(i):

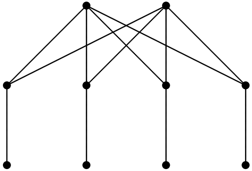


Figure: The graph with sequence $(n, a_4, a_4, a_3, a_2) = (10, 2, 4, 0, 4)$.

$|VG| = 10$(ii):

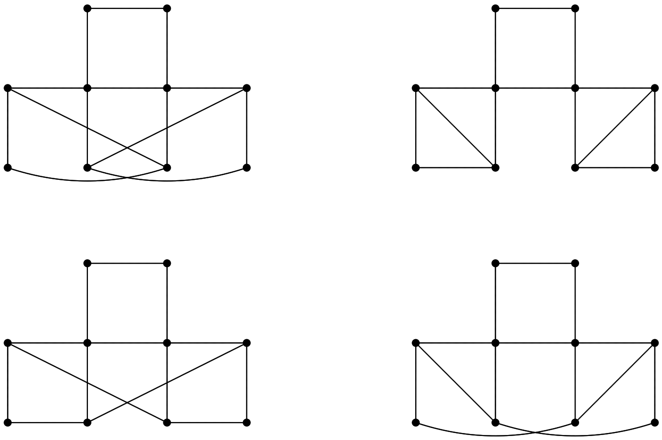


Figure: Graphs with sequence $(n, a_4, a_4, a_3, a_2) = (10, 2, 4, 4, 0)$.

References

[2011CPZ] Y. Chen and R. Pan, and X. Zhang, Two sharp upper bounds for the signless Laplacian spectral radius of graphs, *Discrete Mathematics, Algorithms and Applications*, 3(2011), 185-191.

[2004D] K.C. Das, A characterization on graphs which achieve the upper bound for the largest Laplacian eigenvalue of graphs, *Linear Algebra and its Applications*, 376(2004), 173-186.

[2014HW] Y. P. Huang and C. W. Weng, Spectral radius and average 2-degree sequence of a graph, *Discrete Mathematics, Algorithms and Applications*, 6(2014).

[2001LP] J.S. Li and Y.L. Pan, De Caen′ s inequality and bounds on the largest Laplacian eigenvalue of a graph, *Linear Algebra and its Applications*, 328(2001), 153-160.

[1998LZ] J.S. Li and X.D. Zhang, On Laplacian eigenvalues of a graph, *Linear Algebra and its Applications*, 285(1998), 305-307.

[1998M] R. Merris, A note on Laplacian graph eigenvalues, *Linear Algebra and its Applications*, 285(1998), 33-35.

[2004Z] X.D. Zhang, Two sharp upper bounds for the Laplacian eigenvalues, *Linear Algebra and its Applications*, 376(2004), 207-213.

Thank you for your attention.