Spectral bounds obtained by reweighting entries in a row of a nonnegative matrix

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Abstract

For a square matrix C, the spectral radius $\rho(C)$ is defined as

$$\rho(C) := \max\{ \ |\lambda| \ \ | \ \lambda \text{ is an eigenvalue of } C\},$$

where $|\lambda|$ is the magnitude of complex number λ. It is well known that

$$0 \leq C \leq C' \quad \Rightarrow \quad \rho(C) \leq \rho(C'),$$

where C' is another square matrix of the same size. Now assume that C' has the the same row-sum sequence of a nonnegative matrix C, C' has a positive eigenvector $v = (v_1, v_2, \ldots, v_n)^T$ with the i-th entry the least (i.e. $v_i \leq v_j$ for all j), and $C'[-|i)$ is the submatrix of C' obtained by deleting the i-th column. We will show that

$$0 \leq C[-|i) \leq C'[-|i) \quad \Rightarrow \quad \rho(C) \leq \rho_r(C'),$$

where $\rho_r(C')$ is the largest real eigenvalue of C' Modifying the proof, we also obtain the dual statement that

$$C[-|i) \geq C'[-|i) \geq 0 \quad \Rightarrow \quad \rho(C) \geq \rho_r(C').$$

Notations

1. When C is a real square matrix, the spectral radius $\rho(C)$ is defined as

$$\rho(C) := \max\{\ |\lambda|\ \mid \lambda \text{ is an eigenvalue of } C\},$$

   where $|\lambda|$ is the magnitude of complex number λ.
2. $\rho_r(C)$ is the largest real eigenvalue of C.
3. For a simple undirected graph G, the spectral radius $\rho(G)$ of G is $\rho(A)$, where A is the adjacency matrix of G.

Perron-Frobenius theorem

Let d_1 be the maximum degree of G. It is well-known as a special case of Perron-Frobenius Theorem that

$$\rho(G) \leq d_1.$$

Our realization of the above upper bound:

$$\begin{pmatrix} 0 & 1 & \cdots & 1 & d_1-(n-2) \\ 1 & 0 & & 1 & d_1-(n-2) \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & d_1-(n-2) \\ 1 & 1 & \cdots & 1 & d_1-(n-1) \end{pmatrix}_{n\times n} \qquad \rightarrow \qquad \left(d_1\right).$$

More notations

1. Let n be the order of G,
2. $(d_1, d_2, \ldots, d_n)$ be the degree sequence in decreasing order and
3. $m = (d_1 + \cdots + d_n)/2$ be the number of edges in G.

Spectral upper bound with the number m of edges

In 1985 [2, Corollary 2.3], Brauldi and Hoffman showed that

$$m \leq k(k-1)/2 \quad \Rightarrow \quad \rho(G) \leq k-1,$$

and in 1987 [3], Stanley generalized it as

$$\rho(G) \leq \frac{-1+\sqrt{1+8m}}{2}.$$

Our realization of the above upper bound:

$$\left(\begin{array}{cccc|c} 0 & 1 & \cdots & 1 & d_1-(n-1) \\ 1 & 0 & & 1 & d_2-(n-1) \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & d_n-(n-1) \\ \hline 1 & 1 & \cdots & 1 & 0-n \end{array}\right) \quad \rightarrow \quad \begin{pmatrix} n-1 & 1 \\ 2m-n(n-1) & -n \end{pmatrix}^T.$$

Spectral upper bound with n, m and d_n

In 1998 [4, Theorem 2], Yuan Hong showed that $\rho(G) \le \sqrt{2m-n+1}$, and in 2001 [5, Theorem 2.3], Hong et al. generalized it as

$$\rho(G) \le \frac{d_n - 1 + \sqrt{(d_n+1)^2 + 4(2m - nd_n)}}{2}.$$

Our realization of the above upper bound:

$$\left(\begin{array}{cccc|c} 0 & 1 & \cdots & 1 & d_1 - (n-2) \\ 1 & 0 & & 1 & d_2 - (n-2) \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & d_{n-1} - (n-2) \\ \hline 1 & 1 & \cdots & 1 & d_n - (n-1) \end{array}\right),$$

$$\rightarrow \begin{pmatrix} n-2 & 1 \\ 2m - d_n - (n-1)(n-2) & d_n - (n-1) \end{pmatrix}^T.$$

Spectral upper bound with d_1 and d_ℓ

In 2004 [6, Theorem 2.2], Jinlong Shu and Yarong Wu showed that

$$\rho(G) \leq \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4(\ell - 1)(d_1 - d_\ell)}}{2}$$

for $1 \leq \ell \leq n$. The special case $\ell = 2$ is reproved by Kinkar Ch. Das in 2011 [7].

Our realization of the above upper bound:

$$\left(\begin{array}{cccc|c} 0 & 1 & \cdots & 1 & d_1 - (\ell - 2) \\ 1 & 0 & & 1 & d_1 - (\ell - 2) \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & d_1 - (\ell - 2) \\ \hline 1 & 1 & \cdots & 1 & d_\ell - (\ell - 1) \end{array}\right)_{\ell \times \ell},$$

$$\rightarrow \quad \begin{pmatrix} \ell - 2 & 1 \\ (\ell - 1)(d_1 - \ell + 2) & d_\ell - (\ell - 1) \end{pmatrix}^T.$$

Spectral upper bound with degree sequence

In 2013 [8, Theorem 1.7], Chia-an Liu and Chih-wen Weng showed that

$$\rho(G) \leq \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4\sum_{i=1}^{\ell-1}(d_i - d_\ell)}}{2}$$

for $1 \leq \ell \leq n$.

Our realization of the above upper bound:

$$\left(\begin{array}{cccc|c} 0 & 1 & \cdots & 1 & d_1 - (\ell - 2) \\ 1 & 0 & & 1 & d_2 - (\ell - 2) \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & d_{\ell-1} - (\ell - 2) \\ \hline 1 & 1 & \cdots & 1 & d_\ell - (\ell - 1) \end{array}\right)_{\ell \times \ell},$$

$$\to \begin{pmatrix} \ell - 2 & 1 \\ \sum_{i=1}^{\ell-1} d_i - (\ell - 1)(\ell - 2) & d_\ell - (\ell - 1) \end{pmatrix}^T.$$

Spectral upper bound with row-sums (diagonals 0)

Let $M = (m_{ij})$ be a nonnegative $n \times n$ matrix with diagonal entries 0, row-sums $r_1 \geq r_2 \geq \ldots \geq r_n$, and $e := \max_{1 \leq i,j \leq n} m_{ij}$. In 2013 [9, Theorem 1.9], Yingying Chen, Huiqiu Lin and Jinlong Shu showed that

$$\rho(M) \leq \frac{r_\ell - e + \sqrt{(r_\ell + e)^2 + 4e \sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2}$$

for $1 \leq \ell \leq n$.

Our realization of the above upper bound:

$$\begin{pmatrix} 0 & e & \cdots & e & r_1 - (\ell-2)e \\ e & 0 & & e & r_2 - (\ell-2)e \\ \vdots & & \ddots & \vdots & \vdots \\ e & e & \cdots & 0 & r_{\ell-1} - (\ell-2)e \\ e & e & \cdots & e & r_\ell - (\ell-1)e \end{pmatrix}_{\ell \times \ell},$$

$$\rightarrow \begin{pmatrix} (\ell-2)e & e \\ \sum_{i=1}^{\ell-1} r_i - (\ell-1)(\ell-2)e & r_\ell - (\ell-1)e \end{pmatrix}^T.$$

Spectral upper bound with row-sums

From the assumptions in the last page, in addition assume $d := \max_{1\le i\le n} m_{ii}$. In 2013 [10, Theorem 2.1], Xing Duan and Bo Zhou showed that

$$\rho(M) \le \frac{r_\ell + d - e + \sqrt{(r_\ell - d + e)^2 + 4e\sum_{i=1}^{\ell-1}(r_i - r_\ell)}}{2}$$

for $1 \le \ell \le n$.

Our realization of the above upper bound:

$$\begin{pmatrix} d & e & \cdots & e & r_1 - (\ell-2)e - d \\ e & d & & e & r_2 - (\ell-2)e - d \\ \vdots & & \ddots & \vdots & \vdots \\ e & e & \cdots & d & r_{\ell-1} - (\ell-2)e - d \\ e & e & \cdots & e & r_\ell - (\ell-1)e \end{pmatrix}_{\ell\times\ell},$$

$$\to \begin{pmatrix} (\ell-2)e + d & e \\ \sum_{i=1}^{\ell-1} r_i - (\ell-1)[(\ell-2)e+d] & r_\ell - (\ell-1)e \end{pmatrix}^T.$$

Notations of matrices

1. $C'[-|n)$ is the submatrix of an $n \times n$ matrix C' obtained by deleting the last column.
2. $C'[\alpha|\beta]$ is the $|\alpha| \times |\beta|$ submatrix of C' obtained by retrieving the entries $(a, b) \in \alpha \times \beta$.
3. $C'(\alpha|\beta)$ is the $(n - |\alpha|) \times (n - |\beta|)$ submatrix of C' obtained by deleting the entries $(a, b) \in \alpha \times \beta$.

Rooted vectors

Since the spectral radius is invariant under a permutation of rows and columns simultaneously, we shall assume $i = n$ in the abstract and give the following definition.

Definition

A column vector $v = (v_1, v_2, \ldots, v_n)^T$ is called rooted if $v_j \geq v_n \geq 0$ for $1 \leq j \leq n - 1$.

Main theorem

If $C = (c_{ij})$ is a nonnegative $n \times n$ matrix and $C' = (c'_{ij})$ is an $n \times n$ matrix such that C' has a positive rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T > 0$ for some positive eigenvalue λ and the following (I)-(II) hold

(I) C and C' have the same row-sum vector, and

(II) $C[-|n) \leq C'[-|n)$,

then

$$\rho(C) \leq \lambda$$

with equality if and only if for the index i with $v_i \neq 0$ and $1 \leq j \leq n-1$,

$$(c'_{ij} - c_{ij})(v'_j - v'_n) = 0, \tag{1}$$

where $v^T = (v_1, v_2, \ldots, v_n)$ is a nonnegative left eigenvector of C for $\rho(C)$.

Proof.

By the assumption (I), we have that $c'_{in} - c_{in} = -\sum_{j=1}^{n-1}(c'_{ij} - c_{ij})$ for $1 \leq i \leq n$. Hence

$$((C' - C)v')_i = \sum_{j=1}^{n}(c'_{ij} - c_{ij})v'_j = \sum_{j=1}^{n-1}(c'_{ij} - c_{ij})(v'_j - v'_n) \geq 0. \quad (2)$$

Here the last inequality uses the assumption (II) and $v'_j - v'_n \geq 0$. This is equivalent to

$$Cv' \leq C'v' = \lambda v'. \quad (3)$$

Multiplying v^T from the left to all terms in (3), we have

$$\rho(C)v^T v' = v^T Cv' \leq v^T C'v' = \lambda v^T v'. \quad (4)$$

Now delete the positive term $v^T v'$ to obtain $\rho(C) \leq \lambda$ and finish the proof of the first statement of the theorem.

Continue the Proof.

Assume that $\rho(C) = \lambda$, so the inequality in (4) is equality. Especially $(Cv')_i = (C'v')_i$ in (3) for any i with $v_i \neq 0$. Hence the inequality in (2) is equality. Thus (1) holds.

Conversely, (1) implies that equalities hold in (2) for those i with $v_i \neq 0$, $(Cv')_i = \lambda v'_i$ in (3), equality holds in (4) and $\rho(C) = \lambda$ sequentially.

Example

$$\rho\left(\begin{array}{c|ccc} 1 & 2 & 7 & 1 \\ \hline 4 & 1 & 2 & 1 \\ 4 & 0 & 3 & 1 \\ 4 & 2 & 2 & 0 \end{array}\right) \quad \leq \quad \lambda\left(\begin{array}{c|ccc} 1 & c_1 & c_2 & 10-c_1-c_2 \\ \hline 4 & c_3 & c_4 & 4-c_3-c_4 \\ 4 & c_5 & c_6 & 4-c_5-c_6 \\ 4 & c_7 & c_8 & 4-c_7-c_8 \end{array}\right)$$

$$= \quad \rho\begin{pmatrix} 1 & 10 \\ 4 & 4 \end{pmatrix} = 9, \qquad (c_i \in \mathbb{R}).$$

Note that

$$C' = \left(\begin{array}{c|ccc} 1 & c_1 & c_2 & 10-c_1-c_2 \\ \hline 4 & c_3 & c_4 & 4-c_3-c_4 \\ 4 & c_5 & c_6 & 4-c_5-c_6 \\ 4 & c_7 & c_8 & 4-c_7-c_8 \end{array}\right)$$

has rooted eigenvector $(v'_1, v'_2, v'_3, v'_4) = (5, 4, 4, 4) > 0$ which has $\{j \mid v'_j \neq v'_4\} = \{1\}$. Hence the inequality $\leq$ is an equality.

The dual theorem

If $C = (c_{ij})$ is a nonnegative $n \times n$ matrix and $C' = (c'_{ij})$ is an $n \times n$ matrix such that C' has a positive rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T > 0$ for some positive eigenvalue λ and the following (I)-(II) hold

(I) C and C' have the same row-sum vector, and

(II) $C[-|n) \geq C'[-|n) \geq 0$,

then

$$\rho(C) \geq \lambda$$

with equality if and only if for the index i with $v_i \neq 0$ and $1 \leq j \leq n-1$,

$$(c'_{ij} - c_{ij})(v'_j - v'_n) = 0, \tag{5}$$

where $v^T = (v_1, v_2, \ldots, v_n)$ is a nonnegative left eigenvector of C for $\rho(C)$.

To apply the main theorem, we need to find a way to construct an $n \times n$ matrix $C' = (c'_{ij})$ which has a positive rooted eigenvector $v' = (v'_1, v'_2, \ldots, v'_n)^T > 0$ for some positive eigenvalue λ

Rooted matrices

Definition

An $n \times n$ matrix $C' = (c'_{ij})$ is called rooted if its first $n-1$ columns and the row-sum vector $(r'_1, r'_2, \ldots, r'_n)^T$ are all rooted.

$$\begin{pmatrix} 0 & 1 & \cdots & 1 & d_1 - (n-1) \\ 1 & 0 & & 1 & d_2 - (n-1) \\ \vdots & & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 0 & d_n - (n-1) \\ 1 & 1 & \cdots & 1 & 0 - n \end{pmatrix} + I$$

Lemma

If C' is a rooted matrix, then $\rho(C') = \rho_r(C')$ and C' has a rooted eigenvector for $\rho_r(C')$. Moreover, if $C'[n|n)$ is positive, then v' is positive.

Equitable quotient

For a partition $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of $\{1, \ldots, n\}$, if

$$\pi_{ab} = \sum_{j \in \pi_b} c'_{ij} \qquad \text{for all } i \in \pi_a$$

then the $\ell \times \ell$ matrix $F(C') = (\pi_{ab})$ is called the equitable partition of C' with respect to Π.

$$F \left(\begin{array}{ccc|cc|cc} 1 & 2 & 3 & 3 & 3 & 6 & 6 \\ 3 & 2 & 1 & 4 & 2 & 8 & 4 \\ 2 & 3 & 1 & 5 & 1 & 9 & 3 \\ \hline 3 & 5 & 6 & 1 & 1 & 3 & 4 \\ 4 & 6 & 4 & 2 & 0 & 4 & 3 \\ \hline 0 & 2 & 2 & 2 & 2 & 3 & 2 \\ 1 & 3 & 0 & 3 & 1 & 1 & 4 \end{array} \right) = \begin{pmatrix} 6 & 6 & 12 \\ 14 & 2 & 7 \\ 4 & 4 & 5 \end{pmatrix}$$

Characteristic matrix of a partition

For a partition $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of $\{1, 2, \ldots, n\}$, let S denote the $n \times \ell$ characteristic matrix of Π.

$$\Pi = \{\{1,2,3\},\{4,5\},\{6,7\}\}$$

$$\Rightarrow \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 3 \\ 3 \end{pmatrix}.$$

Lemma

For a partition $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ of $\{1, 2, \ldots, n\}$ with $n \in \pi_\ell$, and a square matrix C' with an equitable quotient $F(C')$,

$$\begin{aligned} & F(C') \text{ has a positive rooted eigenvector } v \text{ for } \rho_r(F(C')) \\ \Rightarrow\quad & C' \text{ has the positive rooted eigenvector } Sv \text{ for } \rho_r(F(C')). \end{aligned}$$

Moreover $\rho_r(C') \geq \rho_r(F(C'))$.

Main application

Let $\Pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be a partition of $\{1, 2, \ldots, n\}$ with $n \in \pi_\ell$, and C an $n \times n$ nonnegative matrix with row-sums $r_1 \geq r_2 \geq \cdots \geq r_n$. For $1 \leq a \leq \ell$ and $1 \leq b \leq \ell - 1$, choose r'_a, c'_{ab} such that

$$\begin{aligned}
r'_a &= \max_{i \in \pi_a} r_i \\
c'_{ab} &\geq \sum_{j \in \pi_b} c_{ij} \qquad \text{for all } i \in \pi_a \\
c'_{ab} &\geq c'_{\ell,b} > 0 \qquad \text{for } a \neq b \\
c'_{a\ell} &= r'_a - \sum_{j=1}^{\ell-1} c'_{aj}.
\end{aligned}$$

Let $C' = (c'_{ab})_{1 \leq a,b \leq \ell}$. Then

$$\rho(C) \leq \rho_r(C').$$

Example

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right) \le \rho_r \begin{pmatrix} 7 & 6 & {\color{red}24} - 13 \\ 12 & 2 & {\color{red}20} - 14 \\ 4 & 4 & {\color{red}13} - 8 \end{pmatrix}$$

The 7×7 matrix on the left has row-sums ${\color{red}24}, 23, 22, {\color{red}20}, 19, {\color{red}13}, 12$.

If applying equitable quotient to a matrix that majors the above 7×7 matrix, one will find the upper bound

$$\rho_r \begin{pmatrix} 7 & 6 & 12 \\ 12 & 2 & 7 \\ 4 & 4 & 6 \end{pmatrix}$$

which is larger than ours.

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Thank you for your attention.