Quasi-spectral characterizations of graphs

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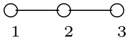
Introduction

The adjacency matrix $A = (a_{ij})$ of G is a binary square matrix of order n with rows and columns indexed by the vertex set VG of G such that for any $i, j \in VG$, $a_{ij} = 1$ if i, j are adjacent in G.



$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The spectral radius $\rho(G)$ of G is the largest eigenvalue of the adjacency matrix A of G.

Let G be a simple connected graph of n vertices and e edges with degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$. The relation between spectral radius $\rho(G)$ and degree sequence d_i (or the number e) of G has been studied by many authors.

- It is well-known that $\rho(G) \leq d_1$ with equality if and only if G is regular [1988Minc, Chapter 2].
- In 1985 [2002BH, Corollary 2.3], Brauldi and Hoffman showed that if $e \leq k(k-1)/2$ then $\rho(G) \leq k-1$ with equality if and only if $G \cong K_n$.
- In 1987 [1987S], Stanley showed that $\rho(G) \leq \frac{-1+\sqrt{1+8e}}{2}$ with equality if and only if $G \cong K_n$.

- In 1998 [1998H, Theorem 2], Yuan Hong showed that $\rho(G) \leq \sqrt{2e-n+1}$ with equality if and only if $G \cong K_{1,n-1}$ or $G \cong K_n$.
- In 2001 [2001HSK, Theorem 2.3], Hong *et al.* showed that $\rho(G) \leq \frac{d_n-1+\sqrt{(d_n+1)^2+4(2e-nd_n)}}{2}$ with equality if and only if G is regular or there exists $2 \leq t \leq n$ such that $d_1 = d_{t-1} = n-1$ and $d_t = d_n$.
- In 2004 [2004SW, Theorem 2.2], Jinlong Shu and Yarong Wu showed that $\rho(G) \leq \frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4(\ell-1)(d_1-d_\ell)}}{2}$ for $1 \leq \ell \leq n$, with equality if and only if G is regular or there exists $2 \leq t \leq \ell$ such that $d_1 = d_{t-1} = n-1$ and $d_t = d_n$.
- The special case $\ell = 2$ of [2004SW, Theorem 2.2] is reproved [2011D].

All the above results can be realized a special case of the following result.

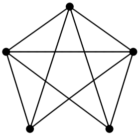
Theorem A

Theorem (Theorem A, [2013LW])

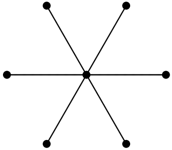
For $1 \leq \ell \leq n$,

$$\rho(G) \leq \phi_\ell := \frac{d_\ell - 1 + \sqrt{(d_\ell + 1)^2 + 4\sum_{i=1}^{\ell-1}(d_i - d_\ell)}}{2}$$

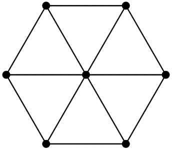
with equality if and only if there exists $1 \leq t \leq \ell$ *such that* $G = K_{t-1} + H$ *for some* $(d_n - t + 1)$*-regular graph* H.



$K_5^- = K_3 + N_2$



$S_7 = K_{1,6} = K_1 + N_6$



$W_7 = K_1 + C_6$

Idea of the proof of Theorem A

Minimize the maximum row-sum of

$$\begin{pmatrix} x_1 & & & & & 0 \\ & \ddots & & & & \\ & & x_{\ell-1} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}^{-1} A \begin{pmatrix} x_1 & & & & & 0 \\ & \ddots & & & & \\ & & x_{\ell-1} & & & \\ & & & 1 & & \\ & & & & \ddots & \\ 0 & & & & & 1 \end{pmatrix}$$

for $x_1, \ldots, x_{\ell-1} \geq 1$, and apply Perron-Frobenius Theorem.

We now assume that G is bipartite.

Applying the idea in the proof of Theorem A, we have a similar theorem for bipartite graph.

Let G be a simple bipartite graph with bipartition orders p and q, and corresponding degree sequences $d_1 \geq d_2 \geq \cdots \geq d_p$ and $d'_1 \geq d'_2 \geq \cdots \geq d'_q$. For $1 \leq s \leq p$ and $1 \leq t \leq q$, let

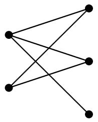
$$\begin{aligned} X_{s,t} &= d_s d'_t + \sum_{i=1}^{s-1}(d_i - d_s) + \sum_{j=1}^{t-1}(d'_j - d'_t), \\ Y_{s,t} &= \sum_{i=1}^{s-1}(d_i - d_s) \cdot \sum_{j=1}^{t-1}(d'_j - d'_t). \end{aligned}$$

Theorem (Theorem B, [2015LW])

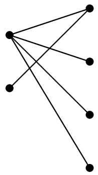
For $1 \le s \le p$ and $1 \le t \le q$, and the notations $X_{s,t}$ and $Y_{s,t}$ in previous page, the spectral radius $\rho(G)$ of a bipartite graph G satisfies

$$\rho(G) \le \phi_{s,t} := \sqrt{\frac{X_{s,t} + \sqrt{X_{s,t}^2 - 4Y_{s,t}}}{2}}.$$

Furthermore, if G is connected then the above equality holds if and only if there exists nonnegative integers $s' < s$ and $t' < t$, and a biregular graph H of bipartition orders $p - s'$ and $q - t'$ respectively such that $G = K_{s',t'} + H$.



$K_{2,3}^5 = {}^5K_{2,3} = K_{1,2} + N_{1,1}$



${}^5K_{2,4} = K_{1,1} + N_{1,3}$

Theorem B looks complex, but it is useful.

As shown in [2015LW], the following are also special cases of Theorem B:

$$\begin{aligned} \rho(G) &\leq \sqrt{e}, \quad \text{[2008BFP]} \\ \rho(G) &\leq \phi_{1,1} = \sqrt{d_1 d'_1}, \quad \text{[2001BZ]} \\ \rho(G) &\leq \phi_{1,q} = \sqrt{e - (q - d_1) d'_q}, \\ \rho(G) &\leq \phi_{p,1} = \sqrt{e - (p - d'_1) d_p}, \\ \rho(G) &\leq \phi_{p,q} = \end{aligned}$$

$$\sqrt{\frac{2e - (pd_p + qd'_q - d_p d'_q) + \sqrt{(pd_p + qd'_q - d_p d'_q)^2 - 4 d_p d'_q (pq - e)}}{2}}.$$

Conjecture C

A. Bhattacharya, S. Friedland and U.N. Peled [2008BFP] gave the Conjecture C below.

Conjecture (Conjecture C)

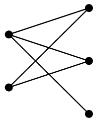
Let $\mathcal{K}(p, q, e)$ denote the family of e-edge subgraphs of the complete bipartite graph $K_{p,q}$ with bipartition orders p and q, and $1 < e < pq$ be integers. An extremal graph that solves

$$\max_{G \in \mathcal{K}(p,q,e)} \rho(G)$$

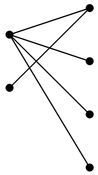
is obtained from a complete bipartite graph by adding one vertex and a corresponding number of edges.

Extremal graphs in Conjecture C

For $e \geq pq - \max(p, q)$ (resp. $e \geq pq - \min(p, q)$), let ${}^{e}K_{p,q}$ (resp. $K^{e}_{p,q}$) denote the graph which is obtained from $K_{p,q}$ by deleting $pq - e$ edges incident on a common vertex in the partite set of order no larger than (resp. no less than) that of the other partite set. Then the extremal graph in Conjecture B is either ${}^{e}K_{s,t}$ or $K^{e}_{s,t}$.



$K^{5}_{2,3} = {}^{5}K_{2,3}$



${}^{5}K_{2,4}$

$$\rho(K^{5}_{2,3}) \geq \rho({}^{5}K_{2,4}) \quad \text{or} \quad \rho({}^{5}K_{2,4}) \geq \rho(K^{5}_{2,3})?$$

Conjecture D

- In 2010 [2010FKSW], Yi-Fan Chen, Hung-Lin Fu, In-Jae Kim, Eryn Stehr and Brendon Watts determined $\rho(K_{p,q}^e)$ and gave an affirmative answer to Conjecture C when $e = pq - 2$.
- Furthermore, they refined Conjecture C for the case when the number of edges is at least $pq - \min(p, q) + 1$ to the following conjecture.

Conjecture (Conjecture D)

Suppose $0 < pq - e < \min(p, q)$. *Then for* $G \in \mathcal{K}(p, q, e)$,

$$\rho(G) \leq \rho(K_{p,q}^e).$$

Conjecture D is affirmative by an application of Theorem B [2015LW].

Average 2-degree sequence

The average 2-degree of the vertex $v \in VG$ is

$$m(v) := \frac{1}{d(v)} \sum_{uv \in EG} d(u),$$

where $d(u)$ is the degree of $u \in VG$.

Let $m_1 \geq m_2 \geq \cdots \geq m_n$ denote the sequence of average 2-degrees of G.

Theorem E

With similar idea of Theorrm A, we showed the following theorem.

Theorem (Theorem E, [2014HW])

For any $b \geq \max\{d(i)/d(j) \mid ij \in EG\}$ *and* $1 \leq \ell \leq n$*,*

$$\rho(G) \leq \frac{m_\ell - b + \sqrt{(m_\ell + b)^2 + 4b\sum_{i=1}^{l-1}(m_i - m_\ell)}}{2},$$

with equality holds iff G *is pseudo regular (i.e.* $m_1 = m_n$*).*

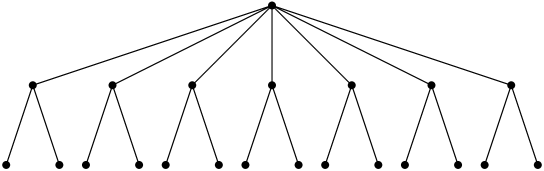


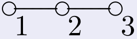
Figure: A pseudo regular tree.

Let $D(G)$=diag$(d(1), d(2), \cdots, d(n))$ be the diagonal matrix, where $d(i)$ is degree of vertex i. Then the matrix

$$L(G) = D(G) - A(G)$$

is called the Laplacian matrix of G.

Example



Then

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

We call the eigenvalues of $L(G)$ the Laplacian eigenvalues of G. It is well known that $L(G)$ is symmetric, positive semidefinite, and every row sum being zero, so we denote the Laplacian eigenvalues of G as in nonincreasing order as

$$\ell_1(G) \geq \ell_2(G) \geq \cdots \geq \ell_n(G) = 0.$$

The Laplacian spread of G is defined as

$$\mathscr{S}_L(G) := \ell_1(G) - \ell_{n-1}(G).$$

G is called strongly regular with parameters (n, k, λ, μ) if G is a k-regular graph with order n and λ (resp. μ) common neighbors of any pair of two adjacent (resp. nonadjacent) vertices.

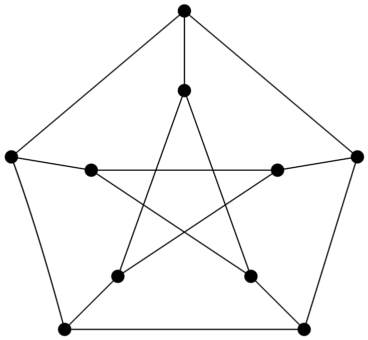


Figure: The Petersen graph is strongly regular with parameters $(10, 3, 0, 1)$.

Let G be a simple connected graph with vertex set $VG = \{1, 2, \cdots, n\}$ and edge set EG. Define

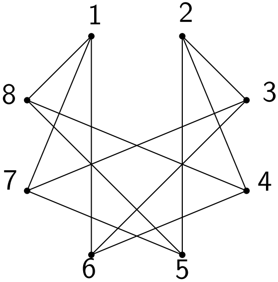
$$\begin{aligned} \lambda_{\min}(G) &:= \min_{ij \in EG} |N(i) \cap N(j)|; \\ \mu_{\min}(G) &:= \min_{ij \notin EG} |N(i) \cap N(j)|, \end{aligned}$$

where $N(i)$ is the neighbor of vertex $i \in V$.

Let δ, Δ be minimum degree and maximum degree of G respectively.

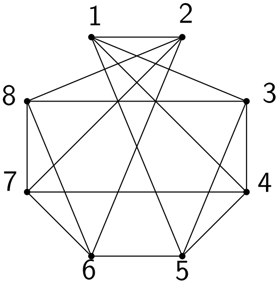
Theorem (Theorem F1, [LWpre])

$$\ell_1(G) \le \frac{2\Delta - \lambda_{\min} + \mu_{\min} + \sqrt{(2\Delta - \lambda_{\min} + \mu_{\min})^2 - 4n\mu_{\min}}}{2}.$$



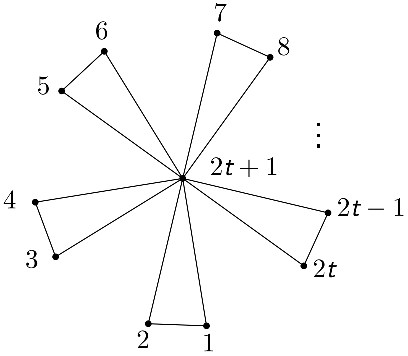
Theorem (Theorem F2, [LWpre])

$$\ell_{n-1}(G) \geq \frac{2\delta - \lambda_{\min} + \mu_{\min} - \sqrt{(2\delta - \lambda_{\min} + \mu_{\min})^2 - 4n\mu_{\min} - 4\delta^2 + 4\Delta^2}}{2}.$$



Theorem (Theorem F3, [LWpre])

$$\mathscr{S}_L(G) \leq \Delta - \delta + \frac{1}{2}\left[\sqrt{(2\Delta - \lambda_{\min} + \mu_{\min})^2 - 4n\mu_{\min}} + \sqrt{(2\delta - \lambda_{\min} + \mu_{\min})^2 - 4n\mu_{\min} - 4\delta^2 + 4\Delta^2}\right].$$

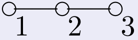


The matrix

$$Q(G) = D(G) - A(G)$$

is called the signless Laplacian matrix of G. Let $q(G)$ denote the largest eigenvalue of $Q(G)$.

Example



Then

$$Q = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Let G be a simple connected graph with vertex set $VG = \{1, 2, \cdots, n\}$ and edge set EG. Define

$$\begin{aligned} \lambda_{\max}(G) &:= \max_{ij \in EG} |N(i) \cap N(j)|; \\ \mu_{\max}(G) &:= \max_{ij \notin EG} |N(i) \cap N(j)|. \end{aligned}$$

Theorem (Theorem G, [FRpre])

$$q(G) \leq \Delta - \frac{\mu_{\max}}{4} + \sqrt{\left(\Delta - \frac{\mu_{\max}}{4}\right)^2 + \Delta(1+\lambda_{\max}) + (n-1)\mu_{\max} - \Delta^2},$$

with equality if and only if G is strongly regular graph with parameters $(n, \Delta, \lambda_{\max}, \mu_{\max})$.

Idea of the proof of Theorem G (Theorems F123 are similar)

Let X be the Perron eigenvector of $Q(G)$. Using the properties that

$$\begin{aligned} X^T Q(G^c) X &\leq 2n-2-q(G), \\ X^T Q(G^c) X &= \sum_{i<j, ij \notin E} (x_i+x_j)^2, \end{aligned}$$

inequalities and some combinatorial arguments to evaluate the term

$$\|A(G)X\|^2 = X^\top A(G)^2 X = \sum_{i \in V} d_i x_i^2 + 2 \sum_{j<k} (A(G)^2)_{jk} x_j x_k. \qquad (1)$$

in

$$\|(q(G)I - D(G))X\|^2 = \|(Q(G)-D(G))X\|^2 = \|A(G)X\|^2,$$

and then obtained an equality with X involved. Try to find another inequality without X.

Future Project

Some graphs are characterized in terms of one of their eigenvalues and some combinatorial parameters.
(Quasi-spectral characterization)

If all eigenvalues of such a graph are provided, it might be possible to characterize the graph only by its eigenvalues.
(Spectral characterization)

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