D-bounded distance-regular graphs

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Let Γ=$(X,\ R)$ denote a finite undirected, connected graph without loops or multiple edges with vertex set X, edge set R, distance function ∂, and diameter D:=max$\{\,\partial(x,y)\mid x,y\in X\}$. For a vertex $x\in X$ and an integer $0\le i\le D$, set $\Gamma_i(x):=\{\,z\in X\mid \partial(x,z)=i\}$. The *valency* $k(x)$ of a vertex $x\in X$ is the cardinality of $\Gamma_1(x)$. The graph Γ is called *regular* (with *valency* k) if each vertex in X has valency k. A graph Γ is said to be *distance-regular* whenever for all integers $0\le h,i,j\le D$, and all vertices $x,y\in X$ with $\partial(x,y)=h$, the number

$$p^h_{ij}=|\Gamma_i(x)\cap\Gamma_j(y)|$$

is independent of x,y. The constants p^h_{ij} are known as the *intersection numbers* of Γ.

From now on let $\Gamma=(X,R)$ be a distance-regular graph with diameter $D\ge 3$. For two vertices $x,y\in X$, with $\partial(x,y)=i$, set

$$\begin{aligned}B(x,y)&:=\Gamma_1(x)\cap\Gamma_{i+1}(y),\\ C(x,y)&:=\Gamma_1(x)\cap\Gamma_{i-1}(y),\\ A(x,y)&:=\Gamma_1(x)\cap\Gamma_{i}(y).\end{aligned}$$

Note that

$$\begin{aligned}|B(x,y)|&=p^i_{1\ i+1},\\ |C(x,y)|&=p^i_{1\ i-1},\\ |A(x,y)|&=p^i_{1\ i}\end{aligned}$$

are independent of x, y. For convenience, set $c_i:=p^i_{1\ i-1}$ for $1\le i\le D$, $a_i:=p^i_{1\ i}$ for $0\le i\le D$, $b_i:=p^i_{1\ i+1}$ for $0\le i\le D-1$ and put $b_D:=0$, $c_0:=0$, $k:=b_0$. Note that k is the valency of Γ.

Recall that a sequence x, z, y of vertices of Γ is *geodetic* whenever

$$\partial(x,z)+\partial(z,y)=\partial(x,y),$$

where ∂ is the distance function of Γ. A sequence x, z, y of vertices of Γ is *weak-geodetic* whenever

$$\partial(x,z)+\partial(z,y)\le\partial(x,y)+1.$$

Definition 1. A subset $\Delta\subseteq X$ is *weak-geodetically closed* if for any weak-geodetic sequence x, z, y of Γ,

$$x,\ y\in\Delta\Longrightarrow z\in\Delta.$$

Weak-geodetically closed subgraphs are called *strongly closed subgraphs* in [6]. If a weak-geodetically closed subgraph Δ of diameter d is regular then it has valency $a_d+c_d=b_0-b_d$, where a_d, c_d, b_0, b_d are intersection numbers of Γ. Furthermore Δ is distance-regular with intersection numbers $a_i(\Delta)=a_i(\Gamma)$ and $c_i(\Delta)=c_i(\Gamma)$ for $1\le i\le d$ [10, Theorem 4.5].

Definition 2. Γ is said to be *i-bounded* whenever for all $x,y\in X$ with $\partial(x,y)\le i$, there is a regular weak-geodetically closed subgraph of diameter $\partial(x,y)$ which contains x and y.

Note that a $(D-1)$-bounded distance-regular graph is clear to be D-bounded. The properties of D-bounded distance-regular graphs were studied in [11], and these properties were used in the classification of classical distance-regular graphs of negative type [12]. Before stating our main result we make one more definition.

By a *parallelogram of length i*, we mean a 4-tuple $xyzw$ consisting of vertices of Γ such that $\partial(x,y)=\partial(z,w)=1$, $\partial(x,w)=i$, and $\partial(x,z)=\partial(y,w)=\partial(y,z)=i-1$. The previous study of parallelogram-free distance-regular graphs can be found in [3, 7, 9]. The following theorem is our main result in this talk.

Theorem 3 *Let Γ denote a distance-regular graph with diameter $D\ge 3$, and intersection numbers $a_1=0$, $a_2\ne 0$. Fix an integer $1\le d\le D-1$ and suppose Γ contains no parallelograms of any length up to $d+1$. Then Γ is d-bounded.*

Theorem 3 is a generalization of [1, Lemma 4.3.13], [4], and is also proved under an additional assumption $c_2>1$ by A. Hiraki [2]. To prove Theorem 3, we need many previous results of [2]. Theorem 3 also answers the problem proposed in [10, p. 299]. Many previous results prove its complement case $a_1 \neq 0$, for examples under an additional assumption $c_2 > 1$ [10] and under the assumptions $a_2 > a_1 > c_2 = 1$ [8]. For the assumptions $a_2 > a_1$ and $c_2 = 1$, H. Suzuki proves the case $d = 2$ in Theorem 3 [8]; in particular Γ contains a regular weak-geodetically closed subgraph Ω of diameter 2. Since the Friendship Theorem [13, Theorem 8.6.39] asserts no such Ω in the case $a_1 = c_2 = 1$, there must be no such distance-regular graph Γ with $a_2 > a_1 = c_2 = 1$ and Γ contains no parallelograms of length 3. Note that the assumption $a_1 \neq 0$ implies $a_2 \neq 0$ [1, Proposition 5.5.1(i)]. Hence Theorem 3 is also true under the weaker assumptions $b_1 > b_2$ and $a_2 \neq 0$. Our method in proving Theorem 3 also works for the case $b_1 > b_2$ and $a_2 \neq 0$ after a slight modification, but we decide not to duplicate the previous works.

On the other hand we suppose that Γ is d-bounded for $d \geq 2$. Let $\Omega \subseteq \Delta$ be two regular weak-geodetically closed subgraphs of diameters $1, 2$ respectively. Since Ω and Δ have different valency $b_0 - b_1$ and $b_0 - b_2$ respectively, we have $b_1 > b_2$. It is also easy to see that Γ contains no parallelograms of any length up to $d + 1$ [10, Lemma 6.5]. With these comments, Theorem 3 is the final step in the following characterization of d-bounded distance-regular graphs in terms of forbidden parallelograms.

Theorem 4. *Let Γ denote a distance-regular graph with diameter $D \geq 3$. Suppose the intersection number $a_2 \neq 0$. Fix an integer $2 \leq d \leq D - 1$. Then the following two conditions (i), (ii) are equivalent:*

(i) Γ is d-bounded.

(ii) Γ contains no parallelograms of any length up to $d + 1$ and $b_1 > b_2$.

Some applications of Theorem 3 were previously given in [2], [5].

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