A matrix realization of spectral bounds

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Abstract

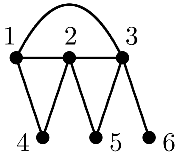
By considering a modified quotient of a nonnegative matrix, we establish an upper bound for its largest real eigenvalue. This is a joint work with Yen-Jen Cheng.

Keywords: nonnegative matrices, spectral radius, spectral bounds

Outline

Preliminaries

Graph, Adjacency Matrix, Degree Sequence



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} \to d_3 = 3 \\ \to d_1 = 4 \\ \to d_2 = 4 \\ \to d_4 = 2 \\ \to d_5 = 2 \\ \to d_6 = 1 \end{matrix}$$

$$n = |VG| = 6, \quad m = |EG| = 8, \quad d_1 \geq d_2 \geq \cdots \geq d_n$$

Notations

Let $C = (c_{ij})$ be an $n \times n$ matrix.

- The **spectral radius** of C is defined to be

$$\rho(C) := \{|\lambda| : \lambda \text{ is an eigenvalue of } C\},$$

  where $|\lambda|$ is the magnitude of complex number λ.
- Let $\rho_r(C)$ denote the largest real eigenvalue of C and $\rho_r(C) = \infty$ if C has no real eigenvalues.

Perron-Frobenius Theorem (PF Theorem)

If C is a nonnegative square matrix, then there exists a nonnegative column vector u such that $Cu = \rho(C)u$.

Examples of matrices that realize existing spectral bounds

Two matrices from PF Theorem

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \leq \rho\begin{pmatrix}0&1&1&1&1&1\\1&0&1&1&1&1\\1&1&0&1&1&1\\1&1&1&0&1&1\\1&1&1&1&0&1\\1&1&1&1&1&0\end{pmatrix}$$

$$= \rho(5) \qquad (1\times 1 \text{ matrix})$$

(equitable quotient)

Stanley, 1987

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \frac{-1+\sqrt{1+8m}}{2} \qquad (m=8) \quad \text{is realized as}$$

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \rho\begin{pmatrix}0&1&1&1&1&1&-2\\1&0&1&1&1&1&-1\\1&1&0&1&1&1&-1\\1&1&1&0&1&1&-3\\1&1&1&1&0&1&-3\\1&1&1&1&1&0&-4\\1&1&1&1&1&1&-6\end{pmatrix} \begin{matrix}\to d_3=3\\\to d_1=4\\\to d_2=4\\\to d_4=2\\\to d_5=2\\\to d_6=1\\\to d_7=0\end{matrix}$$

$$= \rho\begin{pmatrix}5&-14\\1&-6\end{pmatrix} \approx 3.531.$$

Hong et al., 2001

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \frac{d_n-1+\sqrt{(d_n+1)^2+4(2m-nd_n)}}{2}$$ is realized as

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \rho\begin{pmatrix}0&1&1&1&1&-1\\1&0&1&1&1&0\\1&1&0&1&1&0\\1&1&1&0&1&-2\\1&1&1&1&0&-2\\1&1&1&1&1&-4\end{pmatrix} \begin{matrix}\to d_3=3\\ \to d_1=4\\ \to d_2=4\\ \to d_4=2\\ \to d_5=2\\ \to d_6=1\end{matrix}$$

$$(n=6, d_n=1, m=8) \qquad = \rho\begin{pmatrix}4&-5\\1&-4\end{pmatrix} \approx 3.316$$

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Jinlong Shu and Yarong Wu, 2004

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4(\ell-1)(d_1-d_\ell)}}{2}$$ is realized as

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \rho\begin{pmatrix}0&1&1&1&1&0\\1&0&1&1&1&0\\1&1&0&1&1&0\\1&1&1&0&1&0\\1&1&1&1&0&-2\\1&1&1&1&1&-3\end{pmatrix} \begin{matrix}\to d_1=4\\\to d_1=4\\\to d_1=4\\\to d_1=4\\\to d_5=2\\\to d_5=2\end{matrix}$$

$$(\ell=5, d_1=4, d_\ell=2) \quad = \quad \rho\begin{pmatrix}4&-2\\1&-3\end{pmatrix}\approx 3.702.$$

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Liu and $-$, 2013

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \frac{d_\ell-1+\sqrt{(d_\ell+1)^2+4\sum_{i=1}^{\ell-1}(d_i-d_\ell)}}{2}$$ is realized as

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix} \le \rho\begin{pmatrix}0&1&1&1&1&-1\\1&0&1&1&1&0\\1&1&0&1&1&0\\1&1&1&0&1&-2\\1&1&1&1&0&-2\\1&1&1&1&1&-3\end{pmatrix} \begin{matrix}\to d_3=3\\\to d_1=4\\\to d_2=4\\\to d_4=2\\\to d_5=2\\\to d_5=2\end{matrix}$$

$$(\ell=5) \qquad = \rho\begin{pmatrix}4&-5\\1&-3\end{pmatrix}\approx 3.193.$$

Main theorem

Theorem

Let $M = (m_{ab})$ be an $\ell \times \ell$ matrix whose first $\ell - 1$ columns and row-sum vector are all rooted. If $C = (c_{ij})$ is an $n \times n$ nonnegative matrix and there exists a partition $\Pi = (\pi_1, \pi_2, \ldots, \pi_\ell)$ of $[n] := \{1, 2, \ldots, n\}$ such that

$$\max_{i \in \pi_a} \sum_{j \in \pi_b} c_{ij} \le m_{ab} \quad \text{and} \quad \max_{i \in \pi_a} \sum_{j=1}^{n} c_{ij} \le \sum_{c=1}^{\ell} m_{ac}$$

for $1 \le a \le \ell$ and $1 \le b \le \ell - 1$, then $\rho(C) \le \rho_r(M)$.

Example

$$\rho\left(\begin{array}{ccc|cc|cc} 2 & 1 & 3 & 3 & 3 & 12 & 0 \\ 4 & 2 & 1 & 4 & 2 & 6 & 4 \\ 2 & 3 & 1 & 4 & 1 & 8 & 3 \\ \hline 3 & 5 & 3 & 1 & 1 & 3 & 4 \\ 5 & 6 & 1 & 1 & 0 & 3 & 3 \\ \hline 0 & 2 & 1 & 2 & 2 & 6 & 0 \\ 2 & 2 & 0 & 2 & 1 & 1 & 4 \end{array}\right) \le \rho_r\left(\begin{array}{ccc|cc|cc} 2 & 2 & 3 & 3 & 3 & 12 & -1 \\ 4 & 2 & 1 & 4 & 2 & 6 & 5 \\ 2 & 3 & 2 & 4 & 2 & 8 & 3 \\ \hline 4 & 5 & 3 & 1 & 1 & 3 & 3 \\ 5 & 6 & 1 & 1 & 1 & 3 & 3 \\ \hline 1 & 2 & 1 & 2 & 2 & 6 & -1 \\ 2 & 2 & 0 & 2 & 2 & 1 & 4 \end{array}\right)$$

$$= \rho_r\begin{pmatrix} 7 & 6 & 11 \\ 12 & 2 & 6 \\ 4 & 4 & 5 \end{pmatrix}$$

The proof

Key Lemma

Let $C = (c_{ij})$, $C' = (c'_{ij})$, P and Q be $n \times n$ matrices. Assume that

(i) $PCQ \le PC'Q$;

(ii) C' has an eigenvector Qu for λ', where u is a nonnegative column vector and $\lambda' \in \mathbb{R}$;

(iii) C has a left eigenvector $v^T P$ for λ, where v^T i a nonnegative row vector and $\lambda \in \mathbb{R}$; and

(iv) $v^T PQu > 0$.

Then $\lambda \le \lambda'$. Moreover, $\lambda = \lambda'$ if and only if

$$(PC'Q)_{ij} = (PCQ)_{ij} \qquad \text{for } 1 \le i, j \le n \text{ with } v_i \ne 0 \text{ and } u_j \ne 0. \tag{1}$$

Idea of the proof of main theorem

For a given $n \times n$ matrix C, choose suitable $n \times n$ matrix C' such that M is an equitable quotient of C' and by applying $P = I_n$, and

$$Q = I_n + \sum_{i=1}^{n-1} E_{in} = \begin{pmatrix} I_{n-1} & J_{(n-1)\times 1} \\ O_{1\times(n-1)} & 1 \end{pmatrix},$$

we have

$$PCQ \leq PC'Q,$$

so the condition (i) in Key Lemma of previous page is satisfied.

We might need to modify the above chosen C' to let the conditions (ii)-(iv) of Key Lemma be satisfied.

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Applications

$(0,1)$-matrices with a specified number of ones

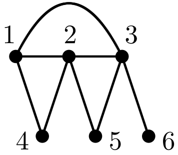
We use the main theorem to identify the unique class of matrices whose largest real eigenvalue is maximum among all $(0,1)$-matrices with a specified number of ones, which is a problem that was posed independently by R. Brualdi and A. Hoffman, as well as F. Friedland, back in 1985.

The above problem restricted to the zero trace situation is considered by Y. Jin and X. Zhang in 2015. We also solve this problem.

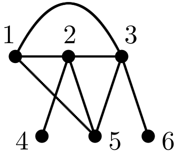
Other P and Q in the Key Lemma

It is interesting to further apply the Key Lemma by choosing other P and Q matrices to solve more conjectures or at least to realize an old result in a new way.

Kelmans transformation



$$G \to G_{4\to5}$$



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

$$G_{4\to5} \cong G_{5\to4}$$

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P. Csikvári, 2009

$$\rho\begin{pmatrix}0&1&1&1&0&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix}\le\rho\begin{pmatrix}0&1&1&0&1&0\\1&0&1&1&1&0\\1&1&0&0&1&1\\0&1&0&0&0&0\\1&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix}$$ is reproved by setting

$P = I_6 + E_{45} = Q^T$ in the Key Lemma to have the condition (i)

$$PAQ=\begin{pmatrix}0&1&1&1&0&0\\1&0&1&2&1&0\\1&1&0&1&1&1\\1&2&1&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix}\le\begin{pmatrix}0&1&1&1&1&0\\1&0&1&2&1&0\\1&1&0&1&1&1\\1&2&1&0&0&0\\1&1&1&0&0&0\\0&0&1&0&0&0\end{pmatrix}=PA_{4\to 5}Q.$$

Use $G_{4\to 5}\cong G_{5\to 4}$ if necessary to have conditions (ii)-(iv).

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