The flipping puzzle on a graph

August 20, 2008

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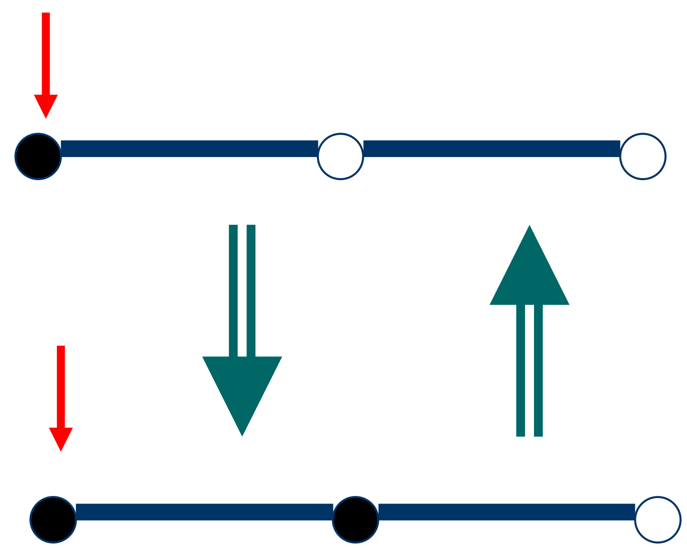
Flipping puzzle

The **configuration** of the flipping puzzle is a fixed simple graph *S*, together with an assignment of state 0(white) or state 1(black) on each vertex of *S*.

A **move** in the puzzle is to select a vertex *s* which has state 1, and then flip the states of all neighbors of *s*.

Example



Equivalent configurations

Two configurations are **equivalent** if one can be obtained from the other by a sequence of moves.

Question: Determine the above equivalent classes (orbits).

Vector representation of configurations

We use $F_2^{\,n}$ to describe the configurations.

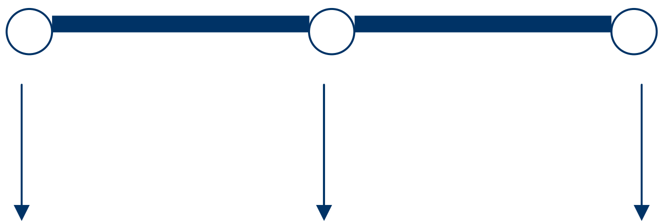
Example:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Matrix representations of moves

Example:



$$x_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad x_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad x_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

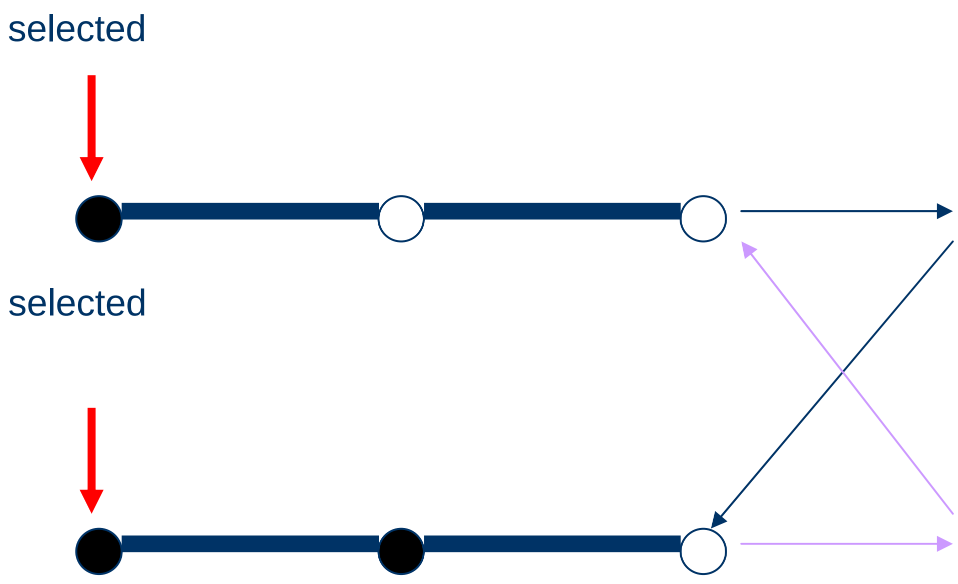
Matrix representations of moves

Definition 0.1. For $s \in S$, we associate a matrix $\mathbf{s} \in \mathrm{Mat}_n(F_2)$, denoted by the bold type of s, as

$$\mathbf{s}_{ab} = \begin{cases} 1, & \text{if } a = b, \text{ or } b = s \text{ and } ab \in R; \\ 0, & \text{else,} \end{cases}$$

where $a, b \in S$ and R is the edge set of S.

Group action



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Flipping groups

Definition 0.2. Let $\mathbf{W}$ denote the subgroup of $\mathrm{GL}_n(F_2)$ generated by the set $\{\mathbf{s} \mid s \in S\}$. $\mathbf{W}$ is referring to the *flipping group* of S.

Theorem (2008, Huang & Weng)

W is isomorphic to W/Z(W),

where W is the Coxeter group of the Dynkin diagram and Z(W) is the center.

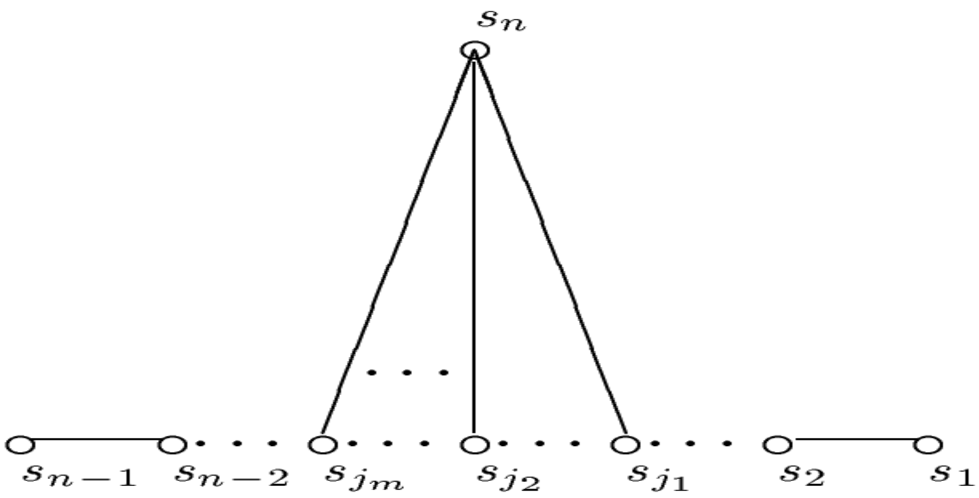
Moreover, |Z(W)| is 1 or 2.

Flipping group

Flipping group is

“the reflection group on a graph”

The graph with a long path



We will determine the orbits for the previous graph.

We need a nice basis to describe our result.

Standard basis

$$\widetilde{s} = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^t$$

where 1 is in the position corresponding to the vertex s

More setting

$$\overline{1} = \widetilde{s}_1,\ \overline{i+1} = \mathbf{s_i s_{i-1}} \cdots \mathbf{s_1} \overline{1} \quad \text{for } 1 \leq i \leq n-1.$$

$$\begin{aligned} \Pi &= \{\overline{1}, \overline{2}, \ldots, \overline{n}\}, \\ \Pi_0 &= \{\overline{i} \in \Pi \mid \ <\overline{i}, \widetilde{s}_n> = 0\}, \\ \Pi_1 &= \Pi - \Pi_0, \end{aligned}$$

Byproduct

Theorem 0.3. *The flipping group* $\mathbf{W}$ *is unique up to isomorphism among all the graphs satisfying Assumption with the given cardinality* $|\Pi_1|$. $\square$

Surprising

Coxeter groups are very different according to different graphs,

but there are at most n-1 non-isomorphic flipping groups in the 2^n graphs that we are concerned.

The Submodule U

Corollary 0.4. *The subspace U spanned by the vectors in Π is a $\mathbf{W}$-submodule of F_2^n.*

Proposition 0.5. *The subspace U in Corollary 0.4 has the basis set*

$$\begin{cases} \Pi, & \text{if } |\Pi_1| \text{ is odd;} \\ \Pi - \{\overline{j}\}, & \text{if } |\Pi_1| \text{ is even} \end{cases}$$

for any $\overline{j} \in \Pi$. Moreover $\widetilde{s}_n \notin U$ if $|\Pi_1|$ is even.

Simple Basis

Set

$$\Delta := \begin{cases} \Pi, & \text{if } |\Pi_1| \text{ is odd;} \\ \Pi \cup \{\overline{n+1}\} - \{\overline{n}\}, & \text{if } |\Pi_1| \text{ is even,} \end{cases}$$

where $\overline{n+1} = \widetilde{s}_n$.

Simple Weight and Weight

Let $\Delta(u)$ be the subset of Δ such that

$$u = \sum_{\bar{i} \in \Delta(u)} \bar{i},$$

set $sw(u) := |\Delta(u)|$, and we refer $sw(u)$ to be the *simple weight* of u.

$$w(u) := |\{s_i \in S \mid u_{s_i} = 1\}|$$

More Setting

For $V \subseteq F_2^n$ and $T \subseteq \{0, 1, \ldots, n\}$,

$$V_T := \{u \in V \mid sw(u) \in T\},$$

and for shortness $V_{t_1,t_2,\ldots,t_i} := V_{\{t_1,t_2,\ldots,t_i\}}$. Let *odd* be the subset of $\{1, 2, \ldots, n\}$ consisting of odd integers.

Three classes of subsets of {1, 2, …, n}

$$A_i := \{j \in [n] \mid j \equiv i, n + |\Pi_1| - i \pmod 4\},$$

$$B_i = \{j \in [n-1] \mid j \equiv i, i+|\Pi_1|-2, n-i, n-i+|\Pi_1|-2 \pmod 4\},$$

$$C_i = \{j \in [n] \mid j \equiv i, i+|\Pi_1|, n+2-i, n+2-i+|\Pi_1| \pmod 4\}.$$

The Orbits (Case 1)

$\lvert\Pi_1\rvert$	n	nontrivial $O \in \mathcal{P}$ (might be repeated)	$\lvert\mathcal{P}\rvert$
$3 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is odd	even	U_{A_j}	3
$3 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is odd	odd	U_{A_j}	4

The Orbits (Case 2)

$\lvert\Pi_1\rvert$	n	nontrivial $O \in \mathcal{P}$ (might be repeated)	$\lvert\mathcal{P}\rvert$
$4 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is even	even	$U_{B_j}, \overline{U}_{C_j}$	6
$4 \leq \lvert\Pi_1\rvert \leq n-3$, $\lvert\Pi_1\rvert$ is even	odd	$U_{B_j}, \overline{U}_{C_j}$	4

The Orbits (Case 3)

$\lvert\Pi_1\rvert$	n	nontrivial $O \in \mathcal{P}$ (might be repeated)	$\lvert\mathcal{P}\rvert$
$\lvert\Pi_1\rvert = 1$		$U_{t,n+1-t}$	$\lceil n+2/2 \rceil$
$\lvert\Pi_1\rvert = 2$	even	$U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}$	$(n+6)/2$
$\lvert\Pi_1\rvert = 2$	odd	$U_{i,n-i}, \overline{U}_{C_1}, \overline{U}_{C_2}$	$(n+3)/2$

The Orbits (Case 4)

$\|\Pi_1\|$	n	nontrivial $O \in \mathcal{P}$ (might be repeated)	$\|\mathcal{P}\|$
$\|\Pi_1\| = n-2$, $\|\Pi_1\|$ is odd	odd	U_{odd}, U_{2i}	$(n+3)/2$
$\|\Pi_1\| = n-2$, $\|\Pi_1\|$ is even	even	$U_{odd}, U_{2h,n-2h}$, $\overline{U}_{odd}, \overline{U}_{2g,n+2-2g}$	$(n+6)/2$

The Orbits (Case 5)

$\lvert\Pi_1\rvert$	n	nontrivial $O \in \mathcal{P}$ (might be repeated)	$\lvert\mathcal{P}\rvert$
$\lvert\Pi_1\rvert = n-1$, $\lvert\Pi_1\rvert$ is odd	even	$U_{2t-1,2t}$	$(n+2)/2$
$\lvert\Pi_1\rvert = n-1$, $\lvert\Pi_1\rvert$ is even	odd	$U_{2h-1,2h,n-2h,,n+1-2h}$, $\overline{U}_{2g-1,2gn+2-2g,n+3-2g}$	$(n+3)/2$

The End!

Thank You for Your Attention!

http://arxiv.org/PS_cache/arxiv/pdf/0808/0808.2104v1.pdf to get the preprint